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A

TREATISE

ON THE

DIFFERENTIAL CALCULUS.

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

By JOHN BAILY, M.A.,
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AND
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DIFFERENTIAL CALCULUS.

SECTION I.

DEFINITIONS AND PRINCIPLES.

1. Of Mathematical quantities considered as constant or variable, arithmetical quantities alone (which are, in fact, merely ratios) are absolutely constant; for since algebraical symbols may be taken to represent any magnitudes whatever, that any one of them represents a constant magnitude can be but *hypothetical*.

Those quantities, however, are termed constant, which remain the same while other quantities connected with them change their value; and these latter quantities are termed variable. Thus in a *given* circle, the radius is a "constant" quantity, while the chord and its distance from the centre are respectively "variable" quantities. Again, in the supposed case of different circles having a common chord, that chord is "constant", and the radius "variable".

"Constant" quantities are most commonly represented by the former letters of the alphabet: "variable" quantities by the latter.

2. Any quantity, the magnitude of which depends, *according to a certain law*, upon the magnitudes of any number of other quantities, is called a "function" of those quantities. Thus, the chord of a circle is a "function" of its distance from the centre, and of the radius of the circle, because its magnitude depends upon the magnitudes of these two quantities

according to a certain law. Again, for the same reason, the space described by a body moving with a constant velocity is a "function" of the velocity and of the time of the body's motion. Also, the volume of a rectangular parallelopiped is a "function" of its three adjacent edges. And each of the quantities x^2, $\sin x$, $\log_{10} x$, is a "function" of x.

3. If a quantity have its magnitude dependant upon the magnitudes of others, some of which are "variable" and some "constant", it is usually said to be a "function" of the variable quantities *only*. Thus, in a *given* circle, the chord would be said to be a "function" of its distance from the centre only. Similarly, if a and b represent "constant", and x and y "variable", quantities, $a^2 + ax - bx^2$ is said to be a "function" of x; $x^3 + axy + by^2$ a "function" of x and y; and so on.

4. When one "variable" quantity is expressed immediately in terms of other "variable" quantities with or without "constants", it is called an "explicit function" of the latter variables. When it is involved with the other "variable" quantities, or appears with them in an unsolved equation, it is called an "implicit function" of the same variables. Thus, if the dependance of y upon x be expressed by the equation $y = \sqrt{2ax - x^2}$, y is an "explicit function" of x: if it be expressed by the equation $y^2 - 2ax + x^2 = 0$, y is an "implicit function" of x.

5. To express *generally* a "function" of any number of quantities, one of the letters f, F, ϕ, ψ is usually prefixed to the letters or symbols representing those quantities, which, when they exceed one in number, are enclosed within brackets, and separated by commas. Thus fx, $F(x,y)$, $\phi \sin x$, $\psi \{\overline{x+h}, \overline{y+k}\}$, respectively represent any function of x; of x and y; of $\sin x$; and of $\overline{x+h}$ and $\overline{y+k}$; any expression with a "vinculum" over it being considered as a *single* symbol.

If two or more of the letters f, F, ϕ, ψ be put before any quantity as x, thus, fFx; this expresses generally the same function of Fx that fx is of x: So that, if we substitute for the general function Fx the particular one $\overline{a+x}$, then fFx becomes $f\overline{a+x}$; and if fx represents x^n, then fFx represents $(a+x)^n$.

If in any part of an investigation any one of the letters, as f, be assumed to indicate a particular function, so that fx represents a particular function of x, as a^x, the letter f will indicate the same function throughout such investigation; and consequently, if fy should occur, it would represent a^y; $f\overline{x+y}$ would represent a^{x+y}; and so on.

Thus, although a function of any number of quantities may be expressed generally by *any* one of the letters f, F, ϕ, ψ prefixed to the symbols representing those quantities in the manner above stated, it is obvious that to express different general functions in the same investigation *different* letters must be used.

6. If $x + h$ be put for x in any function of x represented by fx, the resulting quantity will be the same function of $\overline{x+h}$ as the original quantity is of x, and will be represented by $f\overline{x+h}$. If from $f\overline{x+h}$ the original quantity fx be subtracted, and the "difference" (which it will be convenient to represent by $\Delta_x fx$) be divided by h, there will result a function of two independent quantities x and h, which, on substituting any constant quantity for either of them, as h, becomes a function of the other x. It is also found for many functions of x, and may be so for all, that the quantity represented by $\frac{\Delta_x fx}{h}$ may be reduced to such a form, that when 0 is substituted for h, there still results either a definite function of x or some constant quantity. This last mentioned definite function of x, or constant quantity, when determined, is called "*the differential with regard to x of fx*", and is represented by the symbol $\partial_x fx$, the part ∂_x standing for the expression "*differential with regard to x of*".

For example, if the proposed function of x be x^2, $\Delta_x x^2$ is $(x + h)^2 - x^2$, and $\frac{\Delta_x x^2}{h}$ is $\frac{(x + h)^2 - x^2}{h}$, which may be reduced to the form $\frac{x^2 + 2xh + h^2 - x^2}{h}$ or $2x + h$, and becomes $2x$

when 0 is substituted for h; therefore $\partial_x x^2$ is $2x$; or, as it is usually written, $\partial_x x^2 = 2x$.

Again, let $a + bx$ be the proposed function:

$$\Delta_x (a + bx) \text{ is } a + b(x+h) - (a + bx);$$

$$\therefore \frac{\Delta_x (a+bx)}{h} \text{ is } \frac{a+b(x+h)-(a+bx)}{h} = \frac{bh}{h} = b;$$

$$\therefore \partial_x (a + bx) = b.$$

$$\text{Similarly, } \frac{\Delta_x x}{h} \text{ is } \frac{(x+h) - x}{h} = \frac{h}{h} = 1;$$

$$\therefore \partial_x x = 1.$$

7. Thus it appears that in certain cases quantities represented by $\frac{\Delta_x fx}{h}$ can be made to give functions of x and h, which become definite functions of x or constant quantities when 0 is substituted for h. Since, however, it has not been proved that this is true for all functions of x whatever, *the general propositions which follow, in this Treatise, so far as they relate to the "differentials" of functions, must be considered to extend to those functions only, from which such "differentials" can be derived.*

8. The resulting quantity represented by $\partial_x fx$, when it is a function of x, may evidently itself also have a "differential with regard to x", which, according to the notation, will be represented by $\partial_x \partial_x fx$, or more simply by $\partial_x^2 fx$: and if this latter quantity shall again be a function of x, $\partial_x \partial_x^2 fx$, or $\partial_x^3 fx$ will represent its "differential with regard to x"; and so on.

$$\text{Thus } \partial_x x^2 = 2x, \text{ (Art. 6.)}$$

$$\text{and since } \frac{\Delta_x (2x)}{h} \text{ is } \frac{2(x+h) - 2x}{h}$$

$$= \frac{2h}{h} = 2;$$

$$\therefore \partial_x (2x) = 2; \text{ and } \therefore \partial_x \partial_x x^2 \text{ or } \partial_x^2 x^2 = 2.$$

$\partial_x fx$, $\partial_x^2 fx$, $\partial_x^3 fx$, ... $\partial_x^n fx$ are called respectively the 1st, 2nd, 3rd, ...nth, "differentials with regard to x" of fx; and sometimes the "differentials of the 1st, 2nd, 3rd, ...nth order with regard to x" of fx. But the simple expression, "differential" of any quantity, always means the "1st differential" of that quantity.

9. From the definitions and principles already laid down, it is manifest that the differentials of "constant" quantities have no existence, or may be said to be equal to 0.

Obs. The definitions and preliminary observations with regard to functions of several variables will be found at the beginning of Section vi.

To "differentiate" signifies to perform the process by which "differentials" are obtained.

We proceed to investigate certain general rules for facilitating the differentiation of functions of one variable, and for finding the "differentials" of particular classes of functions.

SECTION II.

GENERAL RULES FOR FACILITATING DIFFERENTIATION, &c.

10. Axiom. If x and h be quantities independent of each other, any equation which subsists between them, or any functions of both or either of them when h is indeterminate, will continue to subsist when 0 is substituted for h, provided that by such substitution no term becomes indefinite, that is, of the form $\frac{a}{0}$, or $\frac{0}{0}$.

11. *If fx and ϕx be functions of x such that $fx = \phi x$ for all values of x, $\partial_x fx = \partial_x \phi x$.*

For since $fx = \phi x$ for all values of x, the equation will hold whatever be substituted for x.

If therefore $x + h$ be substituted for x

$$f\,\overline{x+h} = \phi\,\overline{x+h},$$

for all values of h, whether x be considered to vary or not;

$$\text{hence, } \frac{f\,\overline{x+h} - fx}{h} = \frac{\phi\,\overline{x+h} - \phi x}{h},$$

when h is indeterminate.

But by the hypothesis (Art. 7), the functions of x and h on each side of this last equation are capable of being reduced to such a form that, when 0 is substituted for h, they become definite quantities represented by $\partial_x fx$ and $\partial_x \phi x$. If therefore these quantities so reduced be represented by $F(x,h)$ and $\psi(x,h)$ respectively,

$$F(x,h) = \psi(x,h),$$

when h is indeterminate; and each side of the equation remains definite when 0 is substituted for h, therefore substituting 0 for h, we have, (Art. 10.)

$$\partial_x fx = \partial_x \phi x.$$

12. *If* $F(x, \frac{\Delta_x u}{h}, \frac{\Delta_x v}{h}$, *&c.*$) = 0$ *be an equation which subsists between the quantities* x, $\frac{\Delta_x u}{h}$, *&c., when* h *is indeterminate, the equation will continue to subsist when* 0 *is substituted for* h, *or*

$$F(x, \partial_x u, \partial_x v, \&c.) = 0.$$

For by the hypothesis (Art. 7), the quantities $\frac{\Delta_x u}{h}$, $\frac{\Delta_x v}{h}$, &c. are capable of being reduced to such a form, that when 0 is substituted for h they become definite functions represented by $\partial_x u$, $\partial_x v$, &c.

If therefore these quantities so reduced be represented by $\phi(x, h)$, $\psi(x, h)$, &c.

$$F\{x, \phi(x, h), \psi(x, h), \&c.\} = 0,$$

is an equation which subsists between functions of x and h, when h is indeterminate, and in which no term will assume the form $\frac{a}{0}$ or $\frac{0}{0}$, when 0 is substituted for h.

Hence (Art. 10), when 0 is substituted for h,

$$F\{x, \phi(x, h), \psi(x, h), \&c.\} = 0,$$

$$\text{or, } F(x, \partial_x u, \partial_x v, \&c.) = 0.$$

13. *If* $F(x, y, \frac{\Delta_x u}{h}, \frac{\Delta_y v}{k}$, *&c.*$) = 0$ *be an equation which subsists between* x, y, $\frac{\Delta_x u}{h}$, $\frac{\Delta_y v}{k}$ *&c. when* h *is indeterminate,* v *being a function of* y, *and* y *a function of*

x; and if k be the quantity by which y is increased in consequence of x being increased by h; then when 0 is substituted for h,

$$F(x,\ y,\ \partial_x u,\ \partial_y v,\ \&c.) = 0.$$

For by the hypothesis (Art. 7.) $\frac{\Delta_x u}{h}$ is capable of being reduced to such a form that when 0 is substituted for h, it becomes the definite function $\partial_x u$; and $\frac{\Delta_y v}{k}$ is capable of being reduced to such a form, that when 0 is substituted for k, it becomes the definite function $\partial_y v$; and k is such a function of h, that when $h = 0$, $k = 0$; if therefore k be represented by fh, and $\frac{\Delta_y v}{k}$, when reduced, by $\psi(y, k)$, or $\psi(y, fh)$;

$$F\{x,\ y,\ \phi(x, h),\ \psi(y, fh),\ \&c.\} = 0,$$

is an equation between functions of x and h (for y is a function of x), which subsists when h is indeterminate, and in which no term of the equation assumes the form $\frac{a}{0}$, or $\frac{0}{0}$, when 0 is substituted for h.

Hence (Art. 10), when 0 is substituted for h,

$$F\{x,\ y,\ \phi(x, h),\ \psi(y, fh),\ \&c.\} = 0,$$

$$\text{or,}\quad F(x,\ y,\ \partial_x u,\ \partial_y v,\ \&c.) = 0.$$

14. *Constant quantities, if connected with a function by the signs + or −, are made to disappear by "differentiation": but, if connected by the signs × or ÷ expressed or understood, remain so connected with the "differential" of the function.*

Let fx represent the part of the expression in which the variable x appears, and c the constant part, then

$$\Delta_x(fx \pm c) \text{ is } f(x+h) \pm c - (fx \pm c)$$

$$= f(x+h) - fx \text{ or } \Delta_x fx,$$

$$\therefore\ \frac{\Delta_x(fx \pm c)}{h} = \frac{\Delta_x fx}{h}, \text{ when } h \text{ is indeterminate;}$$

$\therefore$ (12), $\partial_x(fx \pm c) = \partial_x fx$.

Again, $\dfrac{\Delta_x(c.fx)}{h}$ is $\dfrac{c.f\overline{x+h} - c.fx}{h}$

$$= c.\frac{f\overline{x+h} - fx}{h} \quad \text{or } c.\frac{\Delta_x fx}{h};$$

$\therefore$ (12), $\partial_x(c.fx) = c.\partial_x fx$.

Ex. $\partial_x(a + bx) = \partial_x(bx) = b.\partial_x x = b$.

15. If u be a function of y, and y a function of x, u is manifestly a function of x; and the "differential with regard to x" of u might be obtained by first eliminating y, and performing the process of differentiation, as in Art. 6, upon u, which is then expressed immediately in terms of x. This method, however, of finding $\partial_x u$ might be very inconvenient from the difficulty of eliminating y; and it is usually superseded by the following Rule; *viz.*

The "differential with regard to x" of u is equal to the product of the "differential with regard to y" of u and the "differential with regard to x" of y.

To prove this—

Since y is a function of x; if x be increased by h, and k be the quantity by which y is increased in consequence, $k = \Delta_x y$. Again, if u be supposed converted into a function of x by the elimination of y, the "difference" of u produced by substituting $x + h$ for x would be equal to the "difference" of u considered as a function of y produced by substituting $y + k$ for y; and if the former "difference" be represented by $\Delta_x u$, and the latter by $\Delta_y u$, we shall have

$$\Delta_x u = \Delta_y u;$$

$$\therefore\ \frac{\Delta_x u}{h} = \frac{\Delta_y u}{k}.\frac{k}{h} = \frac{\Delta_y u}{k}.\frac{\Delta_x y}{h};$$

$\therefore$ (13), $\partial_x u = \partial_y u.\partial_x y$.

Ex. If $u = ay + y^2$, and $y = x + a$,

then by elimination, $u = ax + a^2 + x^2 + 2ax + a^2$;

$\therefore$ (6) and (14), $\partial_x u = a + 2x + 2a$,

$$= 3a + 2x.$$

Now by Rule, $\partial_x u = \partial_y u . \partial_x y$,

Here $\partial_y u = a + 2y$,

and $\partial_x y = 1$;

$\therefore$ $\partial_x u = a + 2y = 3a + 2x$, as before.

16. The formula proved in Art. 15 is one of very extensive application; and from it may be deduced two others which are frequently useful.

Dividing both sides of the equation $\partial_x u = \partial_y u . \partial_x y$ by $\partial_x y$, we have

$$\partial_y u = \frac{\partial_x u}{\partial_x y}, \text{.....................} \text{(I)},$$

or, *the "differential with regard to y" of u is equal to the "differential with regard to x" of u divided by the "differential with regard to x" of y.*

Again, let $u = x$, then since (Art. 6.) $\partial_x x = 1$, $1 = \partial_y x . \partial_x y$, or

$$\partial_x y = \frac{1}{\partial_y x}, \text{.....................} \text{(II)},$$

or, *the "differential with regard to x" of y is equal to the reciprocal of the "differential with regard to y" of x.*

Ex. Let $y^2 = lx$, to find $\partial_x y$.

$$x = \frac{1}{l} . y^2,$$

$$\therefore \partial_y x = \frac{1}{l} . 2y,$$

$$\therefore \partial_x y = \frac{1}{\partial_y x} = \frac{l}{2y}.$$

or, $= \frac{1}{2}\sqrt{\frac{l}{x}}$; if required in terms of x.

17. If u be a function of v, v of z, z of y, and y of x, then by Art. 15

$$\begin{aligned} \partial_x u &= \partial_v u . \partial_x v \\ &= \partial_v u . \partial_z v . \partial_x z \\ &= \partial_v u . \partial_z v . \partial_y z . \partial_x y. \end{aligned}$$

And the same rule may evidently be continued to any number of connected functions.

If therefore f, F, ϕ, ψ, represent any successive series of single operations upon the variable x producing the complicated function $fF\phi\psi x$, the "differential with regard to x" of this function can be found, if the "differential with regard to x" of each of the simple functions fx, Fx, ϕx, ψx can be found. And since every complicated function must be produced by successive single operations, it follows, that if the differentials with regard to a given variable of all functions representing single operations can be found, the differential with regard to a given variable of any function whatever can also be found.

In a subsequent section the methods of determining the differentials with regard to a given variable of various simple functions representing single operations will be given, the differentials of all complicated functions arising from them being determinable by the above Rule.

18. If u be a function of x, that is, if u and x be connected together by any equation, u and x may be treated as functions of any other arbitrary quantity t, since we are clearly at liberty to assume $x = \phi t$, ϕ and t being respectively any function and quantity whatever, and to consider u as the corresponding function of t, determinable from the equation $x = \phi t$, and the given equation between u and x.

This method of considering quantities which are functions of each other, as a function of some *arbitrary* variable, and of "differentiating" with regard to such variable, will frequently be found convenient, especially in "implicit functions"; because by thus treating the variable quantities in precisely the same manner the symmetry of the resulting equations will be better preserved. Since the variable with regard to which the differentials are taken will in this method of considering them be perfectly arbitrary and undetermined, it will be a proper and convenient mode of distinguishing such differentials to omit the subscribed letter which usually indicates the variable with regard to which the differentials are taken.

19. Def. When the quantity with respect to which the differentials are taken is *arbitrary*, the resulting functions may be termed "differentials" simply, it being unnecessary to take particular notice of the variable of which the original quantities are supposed to be functions. But it is not unusual to express a differential with regard to a given variable by the simple term "differential", when such variable is sufficiently obvious.

20. Since by Art. 15, $\partial_t u = \partial_x u . \partial_t x$, in which t may be any quantity whatever of which u and x are supposed to be functions, by Art. 18,

$$\partial u = \partial_x u . \partial x;$$

or the "differential" of a function of x is equal to the "differential with regard to x" of the same function multiplied by the "differential" of x.

$$\begin{aligned} \text{Thus, if } u &= x^2, \\ \partial_x u &= 2x; \\ \therefore\ \partial u &= 2x.\partial x, \\ \text{or } \partial x^2 &= 2x.\partial x. \end{aligned}$$

Hence, also $\partial_x u = \dfrac{\partial u}{\partial x}$.

or the "differential with regard to x" of u is equal to the "differential" of u divided by the "differential" of x.

Note.—The *fraction* $\dfrac{\partial u}{\partial x}$ must be carefully distinguished from the like *symbol* $\dfrac{du}{dx}$ used in some Treatises to represent the "differential with regard to x" of u. $\partial_x u$ and $\dfrac{\partial u}{\partial x}$ are *equal* (as is here proved), but are by no means *identical*.

21. Since every differential taken with regard to a given variable, as x, is converted into the equivalent one taken with regard to an arbitrary variable by means of the formula $\partial u = \partial_x u \,.\, \partial x$, it is obvious that any equation containing "1st differentials" only, which are taken with regard to an arbitrary variable, may be changed into one in which the variable is given, by merely subscribing that given variable to those symbols of differentiation which are without any subscribed letter, and observing that if it be one of the variables appearing in the equation, as x, $\partial_x x = 1$. Conversely, since $(\partial_x u)^n = \dfrac{(\partial u)^n}{(\partial x)^n}$, it is obvious that any such equation, in which the differentials are taken with regard to a given variable as x, may be changed into one in which such variable is arbitrary by omitting x as a subscribed letter, and multiplying every term by such a power of ∂x as will render the equation homogeneous with respect to the "differentials" in it.

Ex. 1. To convert the equation

$$x.\partial y - y.\partial x = \partial x.\sqrt{x^2 + y^2},$$

into the equivalent one, where the differentials are taken with regard to x.

By the first of the above Rules, the required equation is,

$$x.\partial_x y - y.\partial_x x = \partial_x x.\sqrt{x^2 + y^2},$$

or ($\because \partial_x x = 1$), $\quad x.\partial_x y - y = \sqrt{x^2 + y^2}$.

To prove this fully,

$$\partial y = \partial_x y.\partial x,$$

and substituting this value in the proposed equation,

$$x.\partial_x y.\partial x - y.\partial x = \partial x.\sqrt{x^2 + y^2};$$

$$\therefore\ x.\partial_x y - y = \sqrt{x^2 + y^2}, \text{ as before.}$$

Ex. 2. $a.(\partial_x y)^3 + b.(\partial_x y)^2 + c.\partial_x y + d = 0$; required the equivalent equation when the differentials are taken with regard to an arbitrary variable.

By the second of the above Rules, the required equation is,

$$a.(\partial y)^3 + b.(\partial y)^2.\partial x + c.\partial y.(\partial x)^2 + d.(\partial x)^3 = 0.$$

To prove this fully,

$$\partial_x y = \frac{\partial y}{\partial x},$$

$$\therefore \; a.\left(\frac{\partial y}{\partial x}\right)^3 + b.\left(\frac{\partial y}{\partial x}\right)^2 + c.\frac{\partial y}{\partial x} + d = 0,$$

and $\therefore$ $a.(\partial y)^3+b.(\partial y)^2.\partial x+c.\partial y.(\partial x)^2+d.(\partial x)^3=0$, as before.

22. *To differentiate a quantity consisting of any number of functions of the same variable connected together by the signs + or −.*

Let the quantity be $fx + \phi x - \psi x$ + &c.;

then $\Delta_x(fx+\phi x-\psi x+\&c.)$ is

$$f\overline{x+h} + \phi\overline{x+h} - \psi\overline{x+h} + \&c.$$

$$-(fx + \phi x - \psi x + \&c.)$$

$$=\{f\overline{x+h} - fx\} + \{\phi\overline{x+h} - \phi x\} - \{\psi\overline{x+h} - \psi x\} + \&c.$$

$$= \Delta_x fx + \Delta_x\phi x - \Delta_x\psi x + \&c.$$

$$\therefore \; \frac{\Delta_x(fx + \phi x - \psi x + \&c.)}{h} = \frac{\Delta_x fx}{h} + \frac{\Delta_x\phi x}{h} - \frac{\Delta_x\psi x}{h} + \&c.$$

when h is indeterminate.

$\therefore$ by Art. 12, substituting 0 for h,

$$\partial_x(fx + \phi x - \psi x + \&c.) = \partial_x fx + \partial_x\phi x - \partial_x\psi x + \&c.$$

and considering x to be a function of an arbitrary variable, and therefore fx, ϕx, ψx, &c. functions of the same arbitrary variable,

$$\partial(fx + \phi x - \psi x + \&c.) = \partial fx + \partial\phi x - \partial\psi x + \&c.$$

that is, *the differential of the proposed quantity is found by taking the differentials of the separate functions and retaining the same signs.*

If p, q, r, &c. represent the separate functions,

$$\partial_x(p+q-r+\&c.) = \partial_x p + \partial_x q - \partial_x r + \&c.$$

$$\text{and } \partial(p+q-r+\&c.) = \partial p + \partial q - \partial r + \&c.$$

Ex. $\partial_x(a+bx+cx^2) = \partial_x(bx) + \partial_x(cx^2) = b + 2cx$,

and $\partial(a+bx+cx^2) = b.\partial x + 2cx.\partial x$.

23. *To differentiate a quantity consisting of two functions of the same variable connected by the sign ×.*

Let the quantity be $fx \times \phi x$, or $fx.\phi x$,

then $\Delta_x(fx.\phi x)$ is $f\overline{x+h}.\phi\overline{x+h} - fx.\phi x$

$$= (f\overline{x+h} - fx).\phi\overline{x+h} + (\phi\overline{x+h} - \phi x).fx$$

$$= \Delta_x fx.\phi\overline{x+h} + \Delta_x\phi x.fx;$$

$$\therefore \quad \frac{\Delta_x(fx.\phi x)}{h} = \phi\overline{x+h}.\frac{\Delta_x fx}{h} + fx.\frac{\Delta_x\phi x}{h};$$

when h is indeterminate.

∴ by Art. 12, substituting 0 for h,

$$\partial_x(fx.\phi x) = \phi x.\partial_x fx + fx.\partial_x\phi x.$$

Or, if p and q be the functions

$$\partial_x(pq) = q.\partial_x p + p.\partial_x q;$$

$$\text{and } \partial(pq) = q.\partial p + p.\partial q.$$

that is, *the differential of the product of two functions is equal to the sum of the two quantities formed by multiplying each function by the differential of the other.*

$$\text{Ex.} \quad \partial_x(\overline{a+x}.\overline{b+x}) = \overline{b+x}.\partial_x\overline{a+x} + \overline{a+x}.\partial_x\overline{b+x}$$

$$= b+x+a+x$$

$$= a+b+2x;$$

$$\text{and } \partial(\overline{a+x}.\overline{b+x}) = a.\partial x + b.\partial x + 2x.\partial x.$$

Cor. Since $\partial_x(pq) = q.\partial_x p + p.\partial_x q$,

$$\frac{\partial_x(pq)}{pq} = \frac{\partial_x p}{p} + \frac{\partial_x q}{q};$$

and $$\frac{\partial(pq)}{pq} = \frac{\partial p}{p} + \frac{\partial q}{q}.$$

24. *To differentiate a quantity consisting of any number of functions of the same variable connected by the sign* ×.

Let p, q, r, be three functions of x,

then considering pq as one quantity,

$$\frac{\partial_x(pqr)}{pqr} = \frac{\partial_x(pq)}{pq} + \frac{\partial_x r}{r}$$

$$= \frac{\partial_x p}{p} + \frac{\partial_x q}{q} + \frac{\partial_x r}{r}.$$

And the same law evidently holds whatever be the number of quantities; hence

$$\partial_x(pqr\ldots\ldots) = pqr\ldots\ldots\left\{\frac{\partial_x p}{p} + \frac{\partial_x q}{q} + \frac{\partial_x r}{r} + \ldots\ldots\right\};$$

and $$\partial(pqr\ldots\ldots) = pqr\ldots\ldots\left\{\frac{\partial p}{p} + \frac{\partial q}{q} + \frac{\partial r}{r} + \ldots\ldots\right\}.$$

25. *To differentiate a quantity in the form of a fraction of which the numerator and denominator are functions of the same variable.*

Let $\dfrac{fx}{\phi x}$ be the quantity,

then $\Delta_x \dfrac{fx}{\phi x}$ is $\dfrac{f\overline{x+h}}{\phi\overline{x+h}} - \dfrac{fx}{\phi x}$

$$= \frac{1}{\phi\overline{x+h}.\phi x}\{\phi x.f\overline{x+h} - fx.\phi\overline{x+h}\}$$

$$= \frac{1}{\phi\overline{x+h}.\phi x}\{\phi x.(f\overline{x+h} - fx) - fx(\phi\overline{x+h} - \phi x)\}$$

$$= \frac{1}{\phi\overline{x+h}.\phi x}\{\phi x.\Delta_x fx - fx.\Delta_x\phi x\};$$

$$\therefore\ \frac{\Delta_x \dfrac{fx}{\phi x}}{h} = \frac{1}{\phi\,\overline{x+h}.\phi x}\left\{\phi x.\frac{\Delta_x fx}{h} - fx.\frac{\Delta_x \phi x}{h}\right\};$$

when h is indeterminate;

$\therefore$ by Art. 12. substituting 0 for h,

$$\partial_x \frac{fx}{\phi x} = \frac{1}{(\phi x)^2}\{\phi x.\partial_x fx - fx.\partial_x \phi x\};$$

or, if p and q be the functions,

$$\partial_x \frac{p}{q} = \frac{1}{q^2}\{q.\partial_x p - p.\partial_x q\};$$

$$\text{and } \partial \frac{p}{q} = \frac{1}{q^2}\{q.\partial p - p.\partial q\}.$$

Ex. $$\partial_x \frac{a+x}{b+x} = \frac{1}{(b+x)^2}\{\overline{b+x}.\partial_x \overline{a+x} - \overline{a+x}.\partial_x \overline{b+x}\}$$

$$= \frac{1}{(b+x)^2}\{b+x - \overline{a+x}\}$$

$$= \frac{b-a}{(b+x)^2};$$

$$\text{and } \partial \frac{a+x}{b+x} = \frac{(b-a).\partial x}{(b+x)^2}.$$

Cor. Since $\partial_x \dfrac{p}{q} = \dfrac{1}{q^2}(q.\partial_x p - p.\partial_x q)$;

$$\therefore\ \partial_x \frac{p}{q} = \frac{p}{q}\left(\frac{\partial_x p}{p} - \frac{\partial_x q}{q}\right);$$

$$\text{and } \partial \frac{p}{q} = \frac{p}{q}\left(\frac{\partial p}{p} - \frac{\partial q}{q}\right).$$

26. *To differentiate a quantity in the form of a fraction of which the numerator and denominator consist respectively of products of* several *functions of the same variable.*

Let $\frac{p\,q\,r\ldots}{p_,q_,r_,\ldots}$ be the quantity, in which p, q, r,..., and $p_,$, $q_,$, $r_,$,..., are severally functions of the same variable x. Then, considering $pqr\ldots$, and $p_,q_,r_,\ldots$, as simple quantities.

By Art. 25. Cor.

$$\partial_x\frac{p\,q\,r\ldots}{p_,q_,r_,\ldots}=\frac{p\,q\,r\ldots}{p_,q_,r_,\ldots}\left\{\frac{\partial_x(pqr\ldots)}{pqr\ldots}-\frac{\partial_x(p_,q_,r_,\ldots)}{p_,q_,r_,\ldots}\right\},$$

∴ (24),

$$=\frac{p\,q\,r}{p_,q_,r_,\ldots}\left\{\frac{\partial_x p}{p}+\frac{\partial_x q}{q}+\frac{\partial_x r}{r}+\ldots-\frac{\partial_x p_,}{p_,}-\frac{\partial_x q_,}{q_,}-\frac{\partial_x r_,}{r_,}-\ldots\right\};$$

and

$$\partial\frac{p\,q\,r\ldots}{p_,q_,r_,\ldots}=\frac{p\,q\,r\ldots}{p_,q_,r_,\ldots}\left\{\frac{\partial p}{p}+\frac{\partial q}{q}+\frac{\partial r}{r}+\ldots-\frac{\partial p_,}{p_,}-\frac{\partial q_,}{q_,}-\frac{\partial r_,}{r_,}-\ldots\right\}.$$

Ex.

$$\partial_x\frac{\overline{a+x}.\overline{b+x}}{\overline{a-x}.\overline{b-x}}=\frac{\overline{a+x}.\overline{b+x}}{\overline{a-x}.\overline{b-x}}\left\{\frac{\partial_x\overline{a+x}}{a+x}+\frac{\partial_x\overline{b+x}}{b+x}-\frac{\partial_x\overline{a-x}}{a-x}-\frac{\partial_x\overline{b-x}}{b-x}\right\}$$

$$=\frac{\overline{a+x}.\overline{b+x}}{\overline{a-x}.\overline{b-x}}\left\{\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{a-x}+\frac{1}{b-x}\right\}$$

$$=\frac{1}{(a-x)^2.(b-x)^2}\left\{2\,(a+b)\,(ab-x^2)\right\}.$$

SECTION III.

DIFFERENTIATION OF CERTAIN PARTICULAR CLASSES OF EXPLICIT FUNCTIONS OF ONE VARIABLE.

27. *To differentiate an explicit function of one variable when raised to a constant power.*

Let p represent the function of the variable x, and

1st. Let the index of the power to which it is raised be a positive integer (n); then,

since (24), $\partial_x(p.q.r.....) = pqr.....\left(\frac{\partial_x p}{p} + \frac{\partial_x q}{q} + \frac{\partial_x r}{r} + ...\right)$,

if $p = q = r = ...$and there are n of such functions,

$$\partial_x p^n = p^n.n\frac{\partial_x p}{p}$$
$$= np^{n-1}.\partial_x p.$$

2dly. Let the index of p be a positive fraction, $\frac{m}{n}$;

$$\text{Assume } q = p^{\frac{m}{n}},$$
$$\text{then } q^n = p^m,$$

and since m and n are positive integers, by the first case

$$mp^{m-1}.\partial_x p = \partial_x p^m, \text{ and } nq^{n-1}.\partial_x q = \partial_x q^n;$$

$\therefore$ (11), $nq^{n-1}.\partial_x q = mp^{m-1}.\partial_x p$;

$$\therefore\ \partial_x q = \frac{m}{n}.p^{m-1}.q^{-(n-1)}.\partial_x p,$$

$$\text{or } \partial_x p^{\frac{m}{n}} = \frac{m}{n}.p^{m-1}.p^{-\frac{m}{n}(n-1)}.\partial_x p$$

$$= \frac{m}{n}.p^{\frac{m}{n}-1}.\partial_x p;$$

which is of the same form as when the index is a positive integer.

3dly. Let the index of p be negative;

Since $p^{-n} = \frac{1}{p^n}$,

$$\partial_x p^{-n} = \partial_x \frac{1}{p^n} = -\frac{\partial_x p^n}{p^{2n}}, \text{ Art. 25.}$$

$$= -\frac{np^{n-1}.\partial_x p}{p^{2n}} = -np^{-n-1}.\partial_x p;$$

which is still of the same form.

Hence if n be any integral number or definite fraction, whether positive or negative,

$$\partial_x p^n = np^{n-1}.\partial_x p;$$

$$\text{and } \partial p^n = np^{n-1}.\partial p.$$

that is, *to differentiate a quantity when raised to a constant power, diminish the index by* 1, *and multiply by the index and by the differential of the quantity itself.*

Ex. 1. $\partial_x x^n = nx^{n-1}.\partial_x x = nx^{n-1}$; and $\partial x^n = nx^{n-1}.\partial x$.

Ex. 2. $\partial_x (a + bx^2)^3 = 3(a + bx^2)^2.\partial_x (a + bx^2)$,

$$= 3(a + bx^2)^2.2bx.$$

Ex. 3. $\partial_x \sqrt{a+bx+cx^2} = \partial_x (a+bx+cx^2)^{\frac{1}{2}}$

$$= \tfrac{1}{2}(a+bx+cx^2)^{-\frac{1}{2}}.\partial_x (a+bx+cx^2),$$

$$= \frac{b+2cx}{2\sqrt{a+bx+cx^2}}.$$

Obs. It is particularly useful to remember, that

$$\partial_x \frac{1}{p^n} = -\frac{n.\partial_x p}{p^{n+1}}; \text{ and that } \partial_x \sqrt{p} = \frac{\partial_x p}{2\sqrt{p}}.$$

Ex. $\partial_x \frac{1}{(a^2 + x^2)^n} = -\frac{2nx}{(a^2 + x^2)^{n+1}}.$

28. *To differentiate a quantity consisting of two functions of the same variable, connected by the sign ×, each of which is raised to a constant power.*

Let the quantity be $p^m.q^n$, p and q being functions of x, and m and n not functions of x.

By Art. 23. Cor.

$$\partial(p^m.q^n) = p^m.q^n\left\{\frac{\partial p^m}{p^m} + \frac{\partial q^n}{q^n}\right\}$$

$$= p^m.q^n\left\{\frac{mp^{m-1}.\partial p}{p^m} + \frac{nq^{n-1}.\partial q}{q^n}\right\}$$

$$= p^m.q^n\left\{\frac{m.\partial p}{p} + \frac{n.\partial q}{q}\right\}$$

$$= p^{m-1}.q^{n-1}\left\{mq.\partial p + np.\partial q\right\}.$$

And $\therefore\ \partial_x(p^m.q^n) = p^{m-1}.q^{n-1}\left\{mq.\partial_x p + np.\partial_x q\right\}.$

Ex. 1. $\partial_x\left\{(a+x)^m.(b+x)^n\right\}$

$$= (a+x)^{m-1}.(b+x)^{n-1}.\left\{m(b+x) + n.(a+x)\right\}$$

$$= (a+x)^{m-1}.(b+x)^{n-1}.\left\{mb + na + (m+n)x\right\}.$$

Ex. 2. $\partial_x(\overline{b+x}\cdot\sqrt{a^2+x^2})$

$$= (a^2+x^2)^{-\frac{1}{2}}.\left\{\tfrac{1}{2}(b+x).\partial_x\overline{a^2+x^2} + (a^2+x^2).\partial_x\overline{b+x}\right\}$$

$$= (a^2+x^2)^{-\frac{1}{2}}.\left\{\tfrac{1}{2}(b+x).2x + a^2 + x^2\right\}$$

$$= \frac{a^2+bx+2x^2}{\sqrt{a^2+x^2}}.$$

29. *To differentiate a quantity in the form of a fraction of which the numerator and denominator are functions of the same variable raised to a constant power.*

Let $\frac{p^m}{q^n}$ be the quantity, p and q being functions of x, and m and n not functions of x.

By Art. 25. Cor.

$$\partial\frac{p^m}{q^n} = \frac{p^m}{q^n}\left\{\frac{\partial p^m}{p^m} - \frac{\partial q^n}{q^n}\right\}$$

$$= \frac{p^m}{q^n}\left\{\frac{mp^{m-1}.\partial p}{p^m} - \frac{nq^{n-1}.\partial q}{q^n}\right\}$$

$$= \frac{p^m}{q^n}\left\{\frac{m.\partial p}{p} - \frac{n.\partial q}{q}\right\}$$

$$= \frac{p^{m-1}}{q^{n+1}}\{mq.\partial p - np.\partial q\}.$$

$$\text{and } \partial_x\frac{p^m}{q^n} = \frac{p^{m-1}}{q^{n+1}}\{mq.\partial_x p - np.\partial_x q\}.$$

Ex. 1. $\partial_x\frac{(x+3)^{\frac{5}{2}}}{(x+1)^{\frac{3}{2}}} = \frac{(x+3)^{\frac{3}{2}}}{(x+1)^{\frac{5}{2}}}\left\{\frac{5}{2}(x+1) - \frac{3}{2}(x+3)\right\}$,

[since $\partial_x(x+3) = 1$, and $\partial_x(x+1) = 1$,]

$$= \frac{(x+3)^{\frac{3}{2}}}{(x+1)^{\frac{5}{2}}}\{x-2\}.$$

Ex. 2. $\partial_x\frac{x}{\sqrt{a^2-x^2}} = \frac{1}{(a^2-x^2)^{\frac{3}{2}}}\{a^2-x^2-\tfrac{1}{2}x.\partial_x(a^2-x^2)\}$

$$= \frac{1}{(a^2-x^2)^{\frac{3}{2}}}\{a^2-x^2+x^2\}$$

$$= \frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}.$$

Obs. From $\partial_x(p^m.q^n)$ found in Art. 28, $\partial_x\frac{p^m}{q^n}$ may be at once deduced by substituting $-n$ for n.

Obs. It is useful to commit to memory the formulæ for $\partial_x(p^m.q^n)$, and for $\partial_x\frac{p^m}{q^n}$; since the reduction to the simplest form is effected in the general formulæ, instead of in each particular Example.

SCHOLIUM.

30. If n be a quantity which cannot be represented arithmetically by an integer or definite fraction either positive or negative, as $\sqrt{2}$, $\sin 15^0$, &c., or generally $\sqrt{a}$, $\sin\theta$, &c., a and θ being such that $\sqrt{a}$ and $\sin\theta$ cannot be represented by an integer or definite fraction, the proofs in Arts. 27, 28, 29 will not strictly hold. But as in every calculation in which such quantities occur, the course of proceeding is, to take such an approximation to the actual value of the quantities as may be considered sufficient for the purpose required, which approximation will always be a definite fraction, since a decimal to a given number of places, the usual form in which such an approximation appears, is a definite fraction with the denominator understood, it follows that the above expression for $\partial_x p^n$, as also those which are deduced from it, will be correct, if we consider n as representing that approximation to the true value of the quantity it represents, which is considered sufficiently accurate for the purposes of the calculation or operation in which it occurs. Hence, with this understanding of the meaning of n in such cases, it may be stated generally that for all *possible* values of n, (n being independent of the variable x),

$$\partial_x p^n = np^{n-1}.\partial_x p, \text{ and } \partial p^n = np^{n-1}.\partial p.$$

31. *To differentiate a constant quantity when raised to a power expressed by an explicit function of the variable.*

1st. Let the quantity be a^x,

then $\Delta_x a^x$ is $a^{x+h} - a^x$;

$$\therefore \frac{\Delta_x a^x}{h} = \frac{a^h - 1}{h} . a^x.$$

Now if $\log a$ and $\log b$ represent the logarithms of a and b in a system whose base is any number whatever, or arbitrary,

$$a^h = b^{\frac{\log a}{\log b} . h}, \text{ whatever } a, b, \text{ and } h \text{ are};$$

(see Appendix I, Art. 2. Cor. 2. Snowball's Trigonometry, 3rd Ed.);

$$\therefore \frac{a^h - 1}{h} = \frac{b^{\frac{\log a}{\log b} . h} - 1}{h}$$

$$= \frac{b^{\frac{\log a}{\log b} . h} - 1}{\frac{\log a}{\log b} . h} . \frac{\log a}{\log b};$$

$$\therefore \frac{a^h - 1}{h} : \frac{b^{\frac{\log a}{\log b} . h} - 1}{\frac{\log a}{\log b} . h} = \log a : \log b.$$

It is clear that the two first terms of this proportion become similar functions of a and b respectively, when 0 is put for h. And though in that case each of them appears under the indefinite form $\frac{0}{0}$, yet since they have to each other the definite ratio of $\log a : \log b$, which is not altered by substituting 0 for h, they must be capable of being expressed as *definite* quantities. Therefore, if they be represented by ϕa and ϕb respectively, when 0 is put for h,

$$\phi a : \phi b = \log a : \log b;$$

$$\therefore \frac{\phi a}{\log a} = \frac{\phi b}{\log b} = m,$$

an arithmetical quantity which is the same for all values of a, but depends on the system in which the logarithms are taken; and if ϵ be assumed such that $m = \frac{1}{\log \epsilon}$,

$$\phi a = m.\log a = \frac{\log a}{\log \epsilon},$$

in whatever system the logarithms are taken, since the ratios of logarithms are independent of the system.

But $\frac{\Delta_x a^x}{h} = \frac{a^h - 1}{h}.a^x$, when h is indeterminate;

$\therefore$ substituting 0 for h,

$$\partial_x a^x = \phi a.a^x = \frac{\log a}{\log \epsilon}.a^x;$$

$$\text{and } \partial a^x = \frac{\log a}{\log \epsilon}.a^x.\partial x.$$

2dly. Let the quantity be a^{fx};

$$\text{then } \partial_x a^{fx} = \partial_{fx} a^{fx}.\partial_x fx, \quad (15)$$

$$= \frac{\log a}{\log \epsilon}.a^{fx}.\partial_x fx;$$

$$\text{and } \partial a^{fx} = \frac{\log a}{\log \epsilon}.a^{fx}.\partial fx.$$

Cor. If the logarithms be taken in the system of which the base is ϵ, since $\log_\epsilon \epsilon = 1$,

$$\partial_x a^x = \log_\epsilon a.a^x.$$

$$\text{and } \partial_x a^{fx} = \log_\epsilon a.a^{fx}.\partial_x fx.$$

Hence also, $\partial_x \epsilon^x = \epsilon^x$.

$$\partial \epsilon^x = \epsilon^x.\partial x.$$

$$\partial_x \epsilon^{fx} = \epsilon^{fx}.\partial_x fx.$$

$$\text{And } \partial \epsilon^{fx} = \epsilon^{fx}.\partial fx.$$

Ex. 1. $\partial_x(\epsilon^x.\overline{x-1}) = \epsilon^x.\partial_x \overline{x-1} + \overline{x-1}.\partial_x \epsilon^x$, (23)

$$= \epsilon^x + \overline{x-1}.\epsilon^x$$

$$= \epsilon^x.x.$$

Ex. 2. $$\partial_x \epsilon^{nx} = \epsilon^{nx}.\partial_x (nx)$$
$$= n\epsilon^{nx}.$$

32. *To differentiate the "logarithm" of a function of one variable.*

1st. Let the quantity be $\log_b x$.

$$\text{Then, } x = b^{\log_b x};$$
$$\therefore \partial_x x = \partial_x b^{\log_b x};$$
$$\text{or } 1 = \log_\epsilon b.b^{\log_b x}.\partial_x \log_b x$$
$$= \log_\epsilon b.x.\partial_x \log_b x;$$
$$\therefore \partial_x \log_b x = \frac{1}{\log_\epsilon b}.\frac{1}{x};$$
$$\text{and } \partial \log_b x = \frac{1}{\log_\epsilon b}.\frac{\partial x}{x}.$$

Hence also, $$\partial_x \log_\epsilon x = \frac{1}{\log_\epsilon \epsilon}\ \frac{1}{x}$$
$$= \frac{1}{x};$$
$$\text{and } \partial \log_\epsilon x = \frac{\partial x}{x}.$$

2dly. Let the quantity be $\log_b fx$,

$$\text{then, } \partial_x \log_b fx = \partial_{fx} \log_b fx.\partial_x fx, \quad (15)$$
$$= \frac{1}{\log_\epsilon b}.\frac{1}{fx}.\partial_x fx;$$
$$\text{and } \partial \log_b fx = \frac{1}{\log_\epsilon b}.\frac{\partial fx}{fx};$$

Hence also, $$\partial_x \log_\epsilon fx = \frac{\partial_x fx}{fx};$$
$$\text{and } \partial \log_\epsilon fx = \frac{\partial fx}{fx}.$$

Ex. 1. Required $\partial \log_\epsilon \dfrac{\epsilon^x - 1}{\epsilon^x + 1}$.

Since $\log_\epsilon \dfrac{\epsilon^x - 1}{\epsilon^x + 1} = \log_\epsilon(\epsilon^x - 1) - \log_\epsilon(\epsilon^x + 1)$;

$$\therefore\ \partial \log_\epsilon \frac{\epsilon^x - 1}{\epsilon^x + 1} = \partial \log_\epsilon(\epsilon^x - 1) - \partial \log_\epsilon(\epsilon^x + 1)$$

$$= \frac{\partial(\epsilon^x - 1)}{\epsilon^x - 1} - \frac{\partial(\epsilon^x + 1)}{\epsilon^x + 1};$$

$$= \frac{\epsilon^x.\partial x}{\epsilon^x - 1} - \frac{\epsilon^x.\partial x}{\epsilon^x + 1}$$

$$= \frac{\epsilon^{2x}.\partial x}{\epsilon^{2x} - 1}.$$

Ex. 2. Required $\partial_x \log_b \sqrt{\dfrac{x^2 - a^2}{x^2 + a^2}}$.

Since $\log_b \sqrt{\dfrac{x^2 - a^2}{x^2 + a^2}} = \tfrac{1}{2}\left\{\log_b(x^2 - a^2) - \log_b(x^2 + a^2)\right\}$;

$$\therefore\ \partial_x \log_b \sqrt{\frac{x^2 - a^2}{x^2 + a^2}} = \tfrac{1}{2}.\frac{1}{\log_\epsilon b}\left\{\frac{\partial_x(x^2 - a^2)}{x^2 - a^2} - \frac{\partial_x(x^2 + a^2)}{x^2 + a^2}\right\}$$

$$= \frac{1}{\log_\epsilon b}\left\{\frac{x}{x^2 - a^2} - \frac{x}{x^2 + a^2}\right\}$$

$$= \frac{1}{\log_\epsilon b}.\frac{2a^2 x}{x^2 - a^2}.$$

Ex. 3. Required $\partial\left(\dfrac{1}{\sqrt{b^2 - a^2}}.\log_\epsilon \dfrac{\sqrt{b + a} + \sqrt{b - a}.x}{\sqrt{b + a} - \sqrt{b - a}.x}\right)$.

$$\partial \log_\epsilon \frac{\sqrt{b + a} + \sqrt{b - a}.x}{\sqrt{b + a} - \sqrt{b - a}.x}$$

$$= \partial \log_\epsilon(\sqrt{b + a} + \sqrt{b - a}.x) - \partial \log_\epsilon(\sqrt{b + a} - \sqrt{b - a}.x)$$

$$= \frac{\sqrt{b - a}.\partial x}{\sqrt{b + a} + \sqrt{b - a}.x} + \frac{\sqrt{b - a}.\partial x}{\sqrt{b + a} - \sqrt{b - a}.x}$$

$$= \frac{2\sqrt{b^2 - a^2}.\partial x}{a + b + (a - b)x^2};$$

$\therefore$ the differential required is $\dfrac{2.\partial x}{a + b + (a - b).x^2}$.

33. *To differentiate a quantity which is a function of one variable raised to a power which is also expressed by a function of the same variable.*

Let the quantity be p^q, p and q being functions of the same variable x; then, since

$$\log_\epsilon p^q = q.\log_\epsilon p\,;$$

$$\text{by Art. 11,}\quad \frac{\partial_x p^q}{p^q} = q.\frac{\partial_x p}{p} + \log_\epsilon p.\partial_x q\,;$$

$$\therefore\ \partial_x p^q = p^q\left\{\frac{q.\partial_x p}{p} + \log_\epsilon p.\partial_x q\right\}.$$

$$\text{And}\ \partial p^q = p^q.\left\{\frac{q.\partial p}{p} + \log_\epsilon p.\partial q\right\}.$$

Again, $\partial_x p^{q^r}$ may be deduced from the last result by assuming $q^r = s$; thus

$$\partial_x p^s = p^s.\left\{s.\frac{\partial_x p}{p} + \log_\epsilon p.\partial_x s\right\},$$

$$\therefore\ \partial_x p^{q^r} = p^{q^r}.\left\{q^r.\frac{\partial_x p}{p} + \log_\epsilon p.q^r\left(r.\frac{\partial_x q}{q} + \log_\epsilon q.\partial_x r\right)\right\}$$

$$= p^{q^r}.q^r.\left\{\frac{\partial_x p}{p} + r\log_\epsilon p.\frac{\partial_x q}{q} + \log_\epsilon p.\log_\epsilon q.\partial_x r\right\}.$$

$$\text{And}\ \partial p^{q^r} = p^{q^r}.q^r\left\{\frac{\partial p}{p} + r.\log_\epsilon p.\frac{\partial q}{q} + \log_\epsilon p.\log_\epsilon q.\partial r\right\}.$$

Ex. Required $\partial_x x^x$.

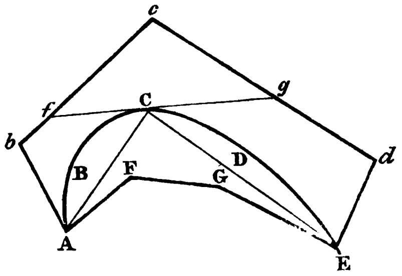
$$\log_\epsilon x^x = x.\log_\epsilon x\,;$$

$$\therefore\ \frac{\partial_x x^x}{x^x} = \log_\epsilon x + x.\frac{1}{x}\,;$$

$$\therefore\ \partial_x x^x = x^x\left\{\log_\epsilon x + 1\right\}.$$

Lemma I. *If a plane curve and a plane rectilinear figure, terminate in the same points, lie in the same plane, have their convexities in the same direction, do not cut each other, and cannot respectively be intersected by a straight* *line in more points than two,* the exterior figure, viz. that which lies on the convex side of the other has the greater perimeter.*

1st. Let $ABCDE$, $AbcdE$, be the curvilinear and rectilinear figures respectively, of which the rectilinear figure $AbcdE$ is the exterior; and if possible let the perimeter $AbcdE$ be not $>$ the perimeter $ABCDE$. Suppose the straight line fCg drawn touching the curve at C and meeting the exterior figure in f and g. Then, since $cf+cg$ is $>gf$, and the perimeter $AbcdE$ is not $>ABCDE$, $AbfgdE$ must be $<ABCDE$. If now we suppose another line drawn touching the curve and terminated by the perimeter of the new rectilinear figure $AbfgdE$, by the same reasoning the perimeter of the corresponding rectilinear figure thus formed will be $<AbfgdE$, and therefore, *à fortiori*, $<ABCDE$. Hence, as the same reasoning may be continued, it appears that the more points are taken in the curve $ABCDE$, and this process repeated, the more the perimeter of the corresponding rectilinear figure recedes from being equal to that of the curve. But it is evident that, on the contrary, it approaches nearer to an equality with it; therefore, the hypothesis that the perimeter $AbcdE$ is not $>ABCD$ is absurd.



2dly. Let $AFGE$ be the rectilinear figure, the curve $ABCDE$ being exterior; and if possible let $AFGE$ be not $<ABCDE$. Take one or more points in the curve, as C, and, by joining each two points that are contiguous, form a rectilinear figure, as ACE, exterior to $AFGE$. Then by producing AF, FG to meet the perimeter ACE it is easily seen that the perimeter ACE is $>AFGE$, and therefore $>ABCDE$; and proceeding as in the first case, it will appear that this second hypothesis is also absurd.

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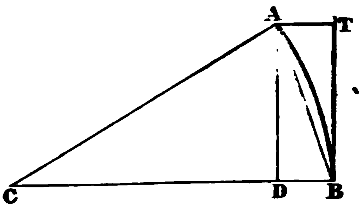
* i. e. Have no "re-entrant angle" as to the rectilinear, and no "contraflexure" or "cusp" as to the curvilinear figure.

Hence the exterior perimeter is in both cases the greater of the two.

Lemma II. *If θ be an angle not $> 90°$,*

$$\frac{\theta}{\sin\theta} = 1 + (\textit{a quantity not} > \tan\frac{\theta}{2} \textit{ and not negative}).$$

Let the angle $ACB\,(=\theta)$ be subtended by the arc AB of a circle described with centre C and radius AC. Draw AD perpendicular to CB, and AT parallel to CB meeting BT which touches the circle at B in T; and join AB.



Then, since the angle ACB is not $> 90°$, AT does not cut the curve AB between the points A and B; $\therefore$ by Lemma I,

$$\text{arc } AB \text{ is not} > AT + TB, \text{ and not} < \text{chord } AB;$$

$$\therefore \text{ arc } AB \text{ is not} > AD + DB, \text{ and not} < AD,$$

$$\text{or } \frac{\text{arc } AB}{AC} \ldots\ldots\ldots \frac{AD}{AC} + \frac{DB}{AC} \ldots\ldots\ldots\ldots \frac{AD}{AC},$$

$$\text{or } \theta \ldots\ldots\ldots \sin\theta + \text{vers}\,\theta \ldots\ldots\ldots\ldots \sin\theta;$$

$$\therefore \frac{\theta}{\sin\theta} \text{ is not} > 1 + \frac{\text{vers}\,\theta}{\sin\theta}, \text{ and not} < 1,$$

$$\ldots\ldots\ldots 1 + \tan\frac{\theta}{2} \ldots\ldots\ldots\ldots\ 1;$$

$$\therefore \frac{\theta}{\sin\theta} = 1 + (\text{a quantity not} > \tan\frac{\theta}{2} \text{ and not negative}).$$

34. *To differentiate goniometrical functions of one variable.*

(1) Required $\partial_x \sin x$.

$$\Delta_x \sin x \text{ is } \sin\overline{x+h} - \sin x;$$

$$\therefore \frac{\Delta_x \sin x}{h} = \frac{\sin \overline{x+h} - \sin x}{h}$$

$$= \frac{2 \sin \frac{h}{2} . \cos \left(x + \frac{h}{2}\right)}{h}$$

$$= \frac{\sin \frac{h}{2}}{\frac{h}{2}} . \cos \left(x + \frac{h}{2}\right),$$

by Lemma II,

$= (1 +$ a quantity not $> \tan \frac{h}{4}$ and not negative$)^{-1} . \cos \left(x + \frac{h}{2}\right)$;

when h is indeterminate;

$\therefore$ substituting 0 for h, in which case the coefficient of

$\cos \left(x + \frac{h}{2}\right)$ becomes equal to 1,

$$\partial_x \sin x = \cos x.$$

$$\text{And } \partial \sin x = \cos x . \partial x.$$

(2) Required $\partial_x \cos x$.

$$\frac{\Delta_x \cos x}{h} = \frac{\cos \overline{x+h} - \cos x}{h}$$

$$= - \frac{2 \sin \frac{h}{2} . \cos \left(x + \frac{h}{2}\right)}{\frac{h}{2}}$$

$$= - \frac{\sin \frac{h}{2}}{\frac{h}{2}} . \sin \left(x + \frac{h}{2}\right)$$

$= - (1 +$ a quantity not $> \tan \frac{h}{4}$ and not negative$)^{-1} . \sin \left(x + \frac{h}{2}\right)$;

when h is indeterminate;

$\therefore$ substituting 0 for h, as before,

$$\partial_x \cos x = - \sin x.$$

$$\text{And } \partial \cos x = - \sin x . \partial x.$$

(3) Required $\partial_x \tan x$.

$$\frac{\Delta_x \tan x}{h} = \frac{\tan \overline{x+h} - \tan x}{h}$$

$$= \frac{\sin \overline{x+h} . \cos x - \cos . \overline{x+h} . \sin x}{h . \cos \overline{x+h} . \cos x}$$

$$= \frac{\sin h}{h . \cos \overline{x+h} . \cos x}$$

$$= \frac{\sin h}{h} . \frac{1}{\cos \overline{x+h} . \cos x}$$

$$= (1 + \text{a quantity not} > \tan \frac{h}{2} \text{ and not neg.})^{-1} . \frac{1}{\cos \overline{x+h} . \cos x};$$

$\therefore$ substituting 0 for h,

$$\partial_x \tan x = \frac{1}{(\cos x)^2} \text{ or } = (\sec x)^2.$$

And $\partial \tan x = (\sec x)^2 . \partial x$.

In a similar manner it may be easily shewn, that

(4) $\partial_x \cot x = - \dfrac{1}{(\sin x)^2}$ or $= - (\operatorname{cosec} x)^2$.

(5) $\partial_x \sec x = \tan x . \sec x$.

(6) $\partial_x \operatorname{cosec} x = - \cot x . \operatorname{cosec} x$.

(7) $\partial_x \operatorname{vers} x = \sin x$.

Each of which may be converted into the corresponding differentials taken with regard to an arbitrary variable by the general rule $\partial u = \partial_x u . \partial x$.

DEF. To express what are called *inverse* goniometrical functions, -1 is prefixed as an index to the corresponding direct function; thus, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, &c. express respectively an angle whose sine, cosine, tangent, &c. is x.

35. *To differentiate inverse goniometrical functions of one variable.*

(1) Required $\partial_x \sin^{-1}x$.

by Def. $\sin \sin^{-1}x = x$,

and, if for convenience of notation $\sin^{-1}x$ be represented by z,

$$\partial_x \sin z = \partial_x x = 1.$$

But (15), $\partial_x \sin z = \partial_z \sin z . \partial_x z$;

$$\therefore\ 1 = \cos z . \partial_x z;$$

$$\therefore\ \partial_x z = \frac{1}{\cos z}$$

$$= \frac{1}{\sqrt{1-(\sin z)^2}};$$

$$\text{or } \partial_x \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$$

(2) Required $\partial_x \cos^{-1}x$.

by Def. $\cos \cos^{-1}x = x$, and if $\cos^{-1}x$ be represented by z,

$$\partial_x \cos z = \partial_x x = 1;$$

$\therefore$ (15), $\partial_z \cos z . \partial_x z = 1$,

or, $-\sin z . \partial_x z = 1$;

$$\therefore\ \partial_x z = -\frac{1}{\sin z}$$

$$= -\frac{1}{\sqrt{1-(\cos z)^2}};$$

$$\text{or } \partial_x \cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}.$$

(3) Required $\partial_x \tan^{-1}x$.

by Def. $\tan \tan^{-1}x = x$, and if $\tan^{-1}x$ be represented by z,

$$\partial_x \tan z = 1;$$

$\therefore$ $\partial_z \tan z . \partial_x z = 1$,

or, $(\sec z)^2 . \partial_x z = 1$;

$$\therefore\ \partial_x z = \frac{1}{(\sec z)^2}$$

$$= \frac{1}{1 + (\tan z)^2};$$

$$\text{or } \partial_x \tan^{-1}x = \frac{1}{1+x^2}.$$

In a similar manner it may easily be shewn that

(4) $$\partial_x \cot^{-1}x = -\frac{1}{1+x^2}.$$

(5) $$\partial_x \sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}.$$

(6) $$\partial_x \operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}.$$

(7) $$\partial_x \operatorname{vers}^{-1}x = \frac{1}{\sqrt{2x-x^2}}.$$

Each of which may be converted into the corresponding differentials taken with regard to an arbitrary variable by the general rule $\partial u = \partial_x u . \partial x$.

36. *To differentiate the more general functions sin fx, cos fx, &c. sin^{-1}fx, cos^{-1}fx, &c.*

By Art. 15, $\partial_x u = \partial_z u . \partial_x z$;

$\therefore$ if for the convenience of notation fx be represented by z,

$$\partial_x \sin z = \partial_z \sin z . \partial_x z$$
$$= \cos z . \partial_x z.$$

(1) $$\text{or } \partial_x \sin fx = \cos fx . \partial_x fx.$$

(2) Similarly, $\partial_x \cos fx = -\sin fx . \partial_x fx.$

(3) $$\partial_x \tan fx = (\sec fx)^2 . \partial_x fx.$$

(4) $$\partial_x \cot fx = -(\operatorname{cosec} fx)^2 . \partial_x fx.$$

(5) $\partial_x \sec fx = \tan fx . \sec fx . \partial_x fx.$

(6) $\partial_x \operatorname{cosec} fx = - \cot fx . \operatorname{cosec} fx . \partial_x fx.$

(7) $\partial_x \operatorname{vers} fx = \sin fx . \partial_x fx.$

Also, in a similar manner it may be shewn that

(1) $\partial_x \sin^{-1} fx = \dfrac{1}{\sqrt{1 - (fx)^2}} . \partial_x fx.$

(2) $\partial_x \cos^{-1} fx = - \dfrac{1}{\sqrt{1 - (fx)^2}} . \partial_x fx.$

(3) $\partial_x \tan^{-1} fx = \dfrac{1}{1 + (fx)^2} . \partial_x fx.$

(4) $\partial_x \cot^{-1} fx = - \dfrac{1}{1 + (fx)^2} . \partial_x fx.$

(5) $\partial_x \sec^{-1} fx = \dfrac{1}{fx . \sqrt{(fx)^2 - 1}} . \partial_x fx.$

(6) $\partial_x \operatorname{cosec}^{-1} fx = - \dfrac{1}{fx . \sqrt{(fx)^2 - 1}} . \partial_x fx.$

(7) $\partial_x \operatorname{vers}^{-1} fx = \dfrac{1}{\sqrt{2fx - (fx)^2}} . \partial_x fx.$

Ex. 1. $\partial_\theta \sin 3\theta = \cos 3\theta . \partial_\theta (3\theta)$

$= 3 . \cos 3\theta.$

Ex. 2. $\partial_x \sin^{-1} \dfrac{x}{a} = \dfrac{1}{\sqrt{1 - \left(\dfrac{x}{a}\right)^2}} . \partial_x \dfrac{x}{a}$

$= \dfrac{1}{\sqrt{a^2 - x^2}}.$

$$\text{Thus also, } \partial_x \cos^{-1}\frac{x}{a} = -\frac{1}{\sqrt{a^2 - x^2}},$$

$$\partial_x \tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2};$$

$$\partial_x \operatorname{vers}^{-1}\frac{x}{a} = \frac{1}{\sqrt{2ax - x^2}}.$$

37. Several of the differentials found in Arts. 34, 35, and 36, might have been *deduced* from others already found: thus

(1) Given $\partial_x \sin x$, to deduce $\partial_x \cos x$.

since, $\cos x = \sin(90^0 - x)$;

$$\partial_x \cos x = \partial_x \sin(90^0 - x),$$

$$(36), \quad = \cos(90^0 - x).\partial_x(90^0 - x),$$

$$= -\sin x; \quad \text{since } \partial_x(90^0 - x) = -\partial_x x = -1.$$

(2) Given $\partial_x \sin^{-1} fx$, to deduce $\partial_x \cos^{-1} fx$.

$$\cos^{-1} fx = 90^0 - \sin^{-1} fx;$$

$$\therefore \partial_x \cos^{-1} fx = -\partial_x \sin^{-1} fx$$

$$= -\frac{1}{\sqrt{1 - (fx)^2}}.\partial_x fx.$$

And the same method is applicable to the other inverse functions which are complements of each other; and accounts for their differentials with regard to the same variable differing from each other in the sign only.

38. Since $\cos^{-1} fx = \sec^{-1}\frac{1}{fx}$, $\tan^{-1} fx = \cot^{-1}\frac{1}{fx}$, $\operatorname{cosec}^{-1} fx = \sin^{-1}\frac{1}{fx}$, &c., it is sometimes useful to substitute one of these equal quantities for the other before the differentiation is commenced; for $\frac{1}{fx}$ is often a simpler quantity than fx.

Ex. Required $\partial_x \sec^{-1}\dfrac{1}{2x^2-1}$.

$$\sec^{-1}\frac{1}{2x^2-1} = \cos^{-1}(2x^2-1).$$

And since $\partial_x \cos^{-1} fx = \dfrac{-1}{\sqrt{1-(fx)^2}}.\partial_x fx$;

$$\therefore\ \partial_x \sec^{-1}\frac{1}{2x^2-1} = \frac{-1}{\sqrt{1-(2x^2-1)^2}}.\partial_x (2x^2-1),$$

$$= \frac{-4x}{\sqrt{4x^2-4x^4}}$$

$$= -\frac{2}{\sqrt{1-x^2}}.$$

SUCCESSIVE DIFFERENTIATION.

39. *To find the* SUCCESSIVE *differentials with regard to any variable of explicit functions of that variable.*

It has been already observed (Art. 8), that the "differential", (or, as it must in this part of the subject be termed for distinction the "*first* differential",) with regard to a given variable of an explicit function is frequently a function of the same variable, and therefore may itself be differentiated; and so produce what is called the "second differential" of the function with regard to that variable. This "second differential", if it be a function of the given variable, may in like manner be differentiated, and produce the "third differential" of the original function with regard to that variable; and the same may be repeated as often as such differential continues to be a function of the variable, with regard to which the differentials are taken.

This may be rendered clear by a few Examples.

Ex. 1. *Required the successive differentials with regard to x of ax^4.*

$$\partial_x(ax^4) = 4ax^3; \text{ (Arts. 14, and 27),}$$

$$\partial_x^2(ax^4) = \partial_x(4ax^3) = 3 \times 4ax^2;$$

$$\partial_x^3(ax^4) = \partial_x(3 \times 4ax^2) = 2 \times 3 \times 4ax;$$

$$\partial_x^4(ax^4) = \partial_x(2 \times 3 \times 4ax) = 1 \times 2 \times 3 \times 4a = 24a.$$

And as $\partial_x^4(ax^4)$ is not a function of x, (Art. 9) $\partial_x^5(ax^4)$ and all the differentials with regard to x of a higher order have no existence or may be said to be equal to 0.

Ex. 2. *Required the successive differentials with regard to x of $\frac{1}{x}$.*

(Art. 27), $\partial_x \frac{1}{x} = -\frac{1}{x^2}$,

$$\partial_x^2 \frac{1}{x} = \partial_x\left(-\frac{1}{x^2}\right) = \frac{2 \times 1}{x^3},$$

$$\partial_x^3 \frac{1}{x} = \partial_x \frac{2 \times 1}{x^3} = -\frac{3 \times 2 \times 1}{x^4},$$

$$\partial_x^4 \frac{1}{x} = \partial_x\left(-\frac{3 \times 2 \times 1}{x^4}\right) = \frac{4 \times 3 \times 2 \times 1}{x^5}.$$

And as the law is now obvious by which each of these equations is derived from the preceding one, it is evident, that

$$\partial_x^n \frac{1}{x} = (-1)^n . \frac{|\underline{n}}{x^{n+1}};$$

(in which equation n is a positive integer and

$|\underline{n}$ represents $n \times \overline{n-1} \times \overline{n-2} \ldots 3 \times 2 \times 1$.)

Any number of successive differentials with regard to x of the proposed function may be obtained by successively substituting in this equation different numbers for n.

Ex. 3. *Required the successive differentials with regard to x of a^x, and of ϵ^x.*

$$\begin{aligned} &\partial_x a^x = \log_\epsilon a . a^x, \quad \text{(Art. 31.)}; \\ \therefore\ &\partial_x^2 a^x = (\log_\epsilon a)^2 . a^x, \\ &\partial_x^3 a^x = (\log_\epsilon a)^3 . a^x, \\ &\dots\dots = \dots\dots\dots\dots \\ &\partial_x^n a^x = (\log_\epsilon a)^n . a^x \dots\dots\dots\dots\dots\dots \ (1). \end{aligned}$$

Also, since $\log_\epsilon \epsilon = 1$,

$$\partial_x^n \epsilon^x = \epsilon^x \dots\dots\dots\dots\dots\dots\dots\dots \ (2).$$

And any number of successive differentials of the proposed functions may be obtained by successively substituting different numbers for n in (1) and (2).

Ex. 4. *Required the successive differentials with regard to x of $\sin x$, and of $\cos x$.*

$$\begin{aligned} &\partial_x \sin x = \cos x, \quad \text{(Art. 34.)} \\ &\partial_x^2 \sin x = -\sin x, \\ &\partial_x^3 \sin x = -\cos x, \\ &\partial_x^4 \sin x = \sin x, \\ &\dots\dots\dots = \dots\dots \end{aligned}$$

$$\therefore \left. \begin{aligned} &\partial_x^{4r} \sin x = \sin x \\ &\partial_x^{4r+1} \sin x = \cos x \\ &\partial_x^{4r+2} \sin x = -\sin x \\ &\partial_x^{4r+3} \sin x = -\cos x \end{aligned} \right\} \text{for any value of } r \text{ which is a positive integer.}$$

Again, $\partial_x \cos x = -\sin x,$

$$\begin{aligned} &\partial_x^2 \cos x = -\cos x, \\ &\partial_x^3 \cos x = \sin x, \\ &\partial_x^4 \cos x = \cos x, \\ &\dots\dots\dots = \dots\dots \end{aligned}$$

$$\therefore \left. \begin{aligned} &\partial_x^{4r} \cos x = \cos x \\ &\partial_x^{4r+1} \cos x = -\sin x \\ &\partial_x^{4r+2} \cos x = -\cos x \\ &\partial_x^{4r+3} \cos x = \sin x \end{aligned} \right\} \text{for any value of } r \text{ which is a positive integer.}$$

Obs. It is sometimes useful to observe, whether the result arising from the first differentiation can be reduced to a form similar to the original function, and thus the law by which each successive equation is derived from the preceding be made evident from one differentiation only. The following is an example:

Ex. 5. *Required the successive differentials with regard to x of $\epsilon^x.\cos x$.*

$$\begin{aligned}\partial_x(\epsilon^x.\cos x) &= \epsilon^x.\cos x - \epsilon^x.\sin x,\\ &= \epsilon^x(\cos x - \sin x)\\ &= 2^{\frac{1}{2}}.\epsilon^x.\cos(45^0 + x);\end{aligned}$$

Which is of the same form as the proposed function, except as to the constant multiplier $2^{\frac{1}{2}}$.

$$\begin{aligned}\text{Hence, } \partial_x\{\epsilon^x.\cos(45^0+x)\} &= 2^{\frac{1}{2}}.\epsilon^x\{\cos(45^0+\overline{45^0+x})\}\partial_x(45^0+x),\\ &= 2^{\frac{1}{2}}.\epsilon^x.\cos(2\times 45^0 + x)\\ \therefore\ \partial_x^2(\epsilon^x.\cos x) &= 2^{\frac{2}{2}}.\epsilon^x.\cos(2\times 45^0 + x);\\ \partial_x^3(\epsilon^x\cos x) &= 2^{\frac{3}{2}}.\epsilon^x.\cos(3\times 45^0 + x);\\ \dots\dots\dots &= \dots\dots\dots\dots\dots\\ \partial_x^n(\epsilon^x\cos x) &= 2^{\frac{n}{2}}.\epsilon^x.\cos(n\times 45^0 + x).\end{aligned}$$

40. *Required the successive differentials with regard to x of $fx \pm c$, and of $c.fx$.*

(1)
$$\begin{aligned}\partial_x(fx \pm c) &= \partial_x fx, \quad \text{(Art. 14)},\\ \therefore\ \partial_x^2(fx \pm c) &= \partial_x^2 fx,\\ \partial_x^3(fx \pm c) &= \partial_x^3 fx,\\ \dots\dots\dots &= \dots\dots\\ \partial_x^n(fx \pm c) &= \partial_x^n fx.\end{aligned}$$

(2) Also,
$$\begin{aligned}\partial_x(c.fx) &= c.\partial_x fx, \quad \text{(Art. 14)},\\ \partial_x^2(c.fx) &= \partial_x(c.\partial_x fx) = c.\partial_x^2 fx,\\ \partial_x^3(c.fx) &= c.\partial_x^3 fx,\\ \dots\dots\dots &= \dots\dots\\ \partial_x^n(c.fx) &= c.\partial_x^n fx.\end{aligned}$$

41. *To find the successive differentials of a quantity consisting of two functions of the same variable connected together by the sign* ×.

Let p and q be the two functions,

then $\partial(pq) = q.\partial p + p.\partial q$, (Art. 23);

$$\therefore\ \partial^2(pq) = \partial(q.\partial p) + \partial(p.\partial q),\ \text{(Art. 22)}$$

$$= q.\partial^2 p + \partial q.\partial p + p.\partial^2 q + \partial p.\partial q,$$

$$= q.\partial^2 p + 2.\partial q.\partial p + \partial^2 q.p.$$

Thus also, $\partial^3(pq) = q.\partial^3 p + \partial q.\partial^2 p$

$$+ 2.\partial q.\partial^2 p + 2.\partial^2 q.\partial p$$

$$+ \partial^2 q.\partial p + \partial^3 q.p,$$

$$= q.\partial^3 p + 3.\partial q.\partial^2 p + 3.\partial^2 q.\partial p + \partial^3 q.p.$$

Similarly, $\partial^4(pq) = q.\partial^4 p + 4.\partial q.\partial^3 p + 6.\partial^2 q.\partial^2 p + 4.\partial^3 q.\partial p + \partial^4 q.p.$

Thus far it appears that the results follow a similar law to that of the terms of an expanded binomial, the index of which is the order of the differential, such index being necessarily a positive integer. Supposing this law to hold for $\partial^n(pq)$, so that

$$\partial^n(pq) = q.\partial^n p + n.\partial q.\partial^{n-1}p + \frac{n(n-1)}{1.2}.\partial^2 q.\partial^{n-2}p + \ldots$$

$$\ldots + n.\partial^{n-1}q.\partial p + \partial^n q.p.$$

Then

$$\partial^{n+1}(pq) = q.\partial^{n+1}p + \partial q.\partial^n p$$

$$+ n.\partial q.\partial^n p + n.\partial^2 q.\partial^{n-1}p$$

$$+ \frac{n(n-1)}{1.2}.\partial^2 q.\partial^{n-1}p + \frac{n(n-1)}{1.2}.\partial^3 q.\partial^{n-2}p$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ n.\partial^{n-1}q.\partial^2 p + n.\partial^n q.\partial p$$

$$+ \partial^n q.\partial p + \partial^{n+1}q.p$$

$$= q.\partial^{n+1}p + (n+1).\partial q.\partial^n p + \frac{(n+1).n}{1.2}.\partial^2 q.\partial^{n-1}p + \ldots$$

$$\ldots + (n+1).\partial^n q.\partial p + \partial^{n+1}q.p.$$

Hence it appears that, if the law hold for $\partial^n(pq)$, it will also hold for $\partial^{n+1}(pq)$.

But it has been shewn that the law holds for $\partial^2(pq)$, and $\partial^3(pq)$, therefore it holds for $\partial^4(pq)$; and if for $\partial^4(pq)$, it holds for $\partial^5(pq)$, and therefore generally for $\partial^n(pq)$;

$$\text{or, } \partial^n(pq) = q.\partial^n p + n.\partial q.\partial^{n-1}p + \frac{n.(n-1)}{1.2}.\partial^2 q.\partial^{n-2}p + \ldots$$
$$\ldots + \partial^n q.p.$$

$$\therefore \text{ also, } \partial^n_x(pq) = q.\partial^n_x p + n.\partial_x q.\partial^{n-1}_x p + \frac{n(n-1)}{1.2}\partial^2_x q.\partial^{n-2}_x p + \ldots$$
$$\ldots + \partial^n_x q.p \ldots\ldots. (\mathrm{I}).$$

Ex. Required the n^{th} differential with regard to x of $\epsilon^x.\log_\epsilon x$.

Let ϵ^x be represented by p,

and $\log_\epsilon x$ .... q.

Then, $\partial^n_x p = \partial^{n-1}_x p = \partial^{n-2}_x p = \ldots\ldots = \epsilon^x$, (Art. 39, Ex. 3),

and $\partial_x q = \frac{1}{x}$, (Art. 32),

$$\therefore \partial^2_x q = -\frac{1}{x^2};$$

$$\partial^3_x q = \frac{1 \times 2}{x^3}$$

$$\ldots = \ldots$$

$$\partial^n_x q = \partial^{n-1}_x\left(\frac{1}{x}\right) = (-1)^{n-1}.\frac{\underline{|n-1}}{x^n}, \text{ (Art. 39, Ex. 2).}$$

And substituting these values in equation (I),

$$\partial^n_x(\epsilon^x.\log_\epsilon x) = \epsilon^x\left\{\log_\epsilon x + n.\frac{1}{x} - n.(n-1).\frac{1}{2x^2} + n(n-1)(n-2).\frac{1}{3x^3}\ldots\right.$$
$$\left.\ldots + (-1)^{n-1}.\underline{|n}.\frac{1}{nx^n}\right\}$$

SECTION IV.

DIFFERENTIATION OF IMPLICIT FUNCTIONS OF ONE VARIABLE.

42. *If* $f(x, y) = 0$ *be an equation subsisting between* x *and* y, $\partial_x^n f(x, y) = 0$ *is an equation subsisting between* $x, y, \partial_x y, \ldots\ldots \partial_x^n y$; $y, \partial_x y, \partial_x^2 y, \ldots\ldots \partial_x^{n-1} y$ *being treated as functions of* x:—*and* $\partial_y^n f(x, y) = 0$ *is an equation subsisting between* $x, y, \partial_y x, \ldots\ldots \partial_y^n x$; $x, \partial_y x, \partial_y^2 x, \ldots\ldots \partial_y^{n-1} x$ *being treated as functions of* y:—*for any value of* n *which is a positive integer.*

For if y be *any* function of x, as ϕx, and this value be substituted in the function $f(x, y)$, but no other change be made, the resulting function $f(x, \phi x)$ is a function of x only, which, without changing its value, may evidently be put under a variety of forms; and if Fx represent *any* one of these forms

$$Fx = f(x, \phi x),$$

whatever is substituted for x;

$$\therefore\ \partial_x Fx = \partial_x f(x, \phi x),\ \text{(Art. 11)}.$$

But $\partial_x f(x, \phi x)$ may be found by treating ϕx as a single quantity, in which case the result will evidently be the same function of x, ϕx, and $\partial_x \phi x$, that $\partial_x f(x, y)$ would be of x, y, and $\partial_x y$; and since $y = \phi x$ and $\partial_x y = \partial_x \phi x$,

$$\therefore\ \partial_x Fx = \partial_x f(x, y),$$

y being treated as a function of x.

Again, since $\partial_x f(x, y)$ is a function of x, y, and $d_x y$, which may be represented by $\psi(x, y, \partial_x y)$, or, in terms of x, by $\psi(x, \phi x, \partial_x \phi x)$,

$$\partial_x F x = \psi(x, \phi x, \partial_x \phi x),$$

whatever is substituted for x;

and since each side of the equation is a function of x only,

$$\partial_x \partial_x F x = \partial_x \psi(x, \phi x, \partial_x \phi x), \text{ (Art. 11)};$$

and reasoning in the same manner as before,

$$\begin{aligned}\partial_x^2 F x &= \partial_x \psi(x, y, \partial_x y)\\ &= \partial_x \partial_x f(x, y)\\ &= \partial_x^2 f(x, y),\end{aligned}$$

y and $\partial_x y$ being treated as functions of x.

And as the same reasoning may evidently be continued successively to differentials of any order; generally

$$\partial_x^n F x = \partial_x^n f(x, y),$$

y, $\partial_x y \ldots\ldots \partial_x^{n-1} y$ being treated as functions of x.

Now let ϕx be the value of y in terms of x derived from the equation $f(x, y) = 0$; it is evident that if this value of y be substituted in the function $f(x, y)$, the resulting function $f(x, \phi x)$ may be reduced to such a form that every term would be capable of being destroyed by a corresponding term; and if Fx represent this function when so reduced, it is also evident (Art. 11), that $\partial_x F x$, $\partial_x^2 F x \ldots \partial_x^n F x$, must respectively consist of terms which would mutually destroy each other. Hence it follows that these quantities must respectively $= 0$, and consequently

$\partial_x f(x, y)$, $\partial_x^2 f(x, y) \ldots\ldots \partial_x^n f(x, y)$ must respectively $= 0$,

y, $\partial_x y, \ldots \partial_x^{n-1} y$ being treated as functions of x.

By similar reasoning it would follow that

$\partial_y f(x,y)$, $\partial_y^2 f(x,y)$......$\partial_y^n f(x,y)$ must respectively $= 0$,

x, $\partial_y x$......$\partial_y^{n-1} x$ being treated as functions of y.

Obs. As in all these equations the value of x or y is perfectly arbitrary, it is unnecessary to consider them as representing the same quantities throughout the equations; though if any elimination or other operation is to be performed between any two or more of the equations, it is necessary that the same symbols should represent the same quantities.

43. *To find the successive differentials with regard to x of an implicit function of x.*

If y be the function, and the relation between x and y be given by the equation $f(x,y) = 0$, it has been shewn (Art. 42), that

$$\partial_x f(x,y) = 0, \quad \partial_x^2 f(x,y) = 0 \ldots\ldots \partial_x^n f(x,y) = 0,$$

the quantities $\partial_x f(x,y)$, $\partial_x^2 f(x,y) \ldots \partial_x^n f(x,y)$, being respectively functions of x, y, and the differentials with regard to x of y up to the order which is the same as the order of the differential of $f(x,y)$.

Hence the value of $\partial_x y$ in terms of x may be found by considering x and consequently y to have the same value in the two equations

$$f(x,y) = 0, \quad \partial_x f(x,y) = 0,$$

eliminating y between them, and then solving the resulting equation with regard to $\partial_x y$.

Similarly, the value of $\partial_x^2 y$ in terms of x may be found from the equations

$$f(x,y) = 0, \quad \partial_x f(x,y) = 0, \quad \partial_x^2 f(x,y) = 0,$$

by eliminating y, and $\partial_x y$, and solving the resulting equation with regard to $\partial_x^2 y$.

And generally, $\partial_x^n y$ may be found in like manner from the $n+1$ equations

$$f(x,y) = 0, \quad \partial_x f(x,y) = 0 \ldots\ldots \partial_x^n f(x,y) = 0.$$

Obs. It is evident from the mode in which it is obtained that $\partial^n_x f(x,y) = 0$ is a simple equation with regard to $\partial^n_x y$, and consequently that the required quantity $\partial^n_x y$ can always be found in terms of x, y, and the inferior quantities $\partial^{n-1}_x y$, $\partial^{n-2}_x y \ldots \ldots \partial_x y$; and since the same is true for each of the equations for determining such inferior quantities, it follows, that the required quantity $\partial^n_x y$ can always be found by successive substitutions in terms of x and y; but, whether it can be found in terms of x, or y, only, will depend upon the possibility of solving the original equation $f(x,y) = 0$ with regard to x, or y.

Ex. 1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Required $\partial_x y$, and $\partial^2_x y$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

$$\therefore\ \partial_x \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0.$$

$$\text{or,}\ \partial_x \frac{x^2}{a^2} + \partial_x \frac{y^2}{b^2} = 0.$$

Now, $\partial_x \frac{x^2}{a^2} = \frac{2x}{a^2}$, and $\partial_x \frac{y^2}{b^2} = \partial_y \frac{y^2}{b^2}.\partial_x y = \frac{2y}{b^2}.\partial_x y$,

$$\therefore\ \frac{2x}{a^2} + \frac{2y}{b^2}.\partial_x y = 0,$$

and $\partial_x y = -\frac{b^2 x}{a^2 y}$, or $= \pm\frac{b}{a}.\frac{x}{\sqrt{a^2 - x^2}}$, if required in terms of x.

$$\text{Again,}\quad \partial_x\left(\frac{2x}{a^2} + \frac{2y}{b^2}.\partial_x y\right) = 0,$$

$$\text{or,}\quad \frac{1}{a^2} + \frac{1}{b^2}.\partial_x y + \frac{y}{b^2}.\partial^2_x y = 0;$$

$$\therefore\ \frac{1}{a^2} - \frac{x}{a^2 y} + \frac{y}{b^2}.\partial^2_x y = 0;$$

$$\therefore\ \partial^2_x y = \frac{b^2}{a^2}.\left(\frac{x}{y} - 1\right),$$

$$\text{or,}\quad = \pm\frac{b}{a}.\frac{x}{\sqrt{a^2 - x^2}} - \frac{b^2}{a^2}.$$

Ex. 2. $(x-a)^2+(y-b)^2=c^2$. Required $\partial_y x$, and $\partial_y^2 x$.

$$\partial_y\{(x-a)^2+(y-b)^2-c^2\}=0;$$

$$\therefore\ 2(x-a).\partial_y x+2(y-b)=0,$$

$$\therefore\ \partial_y x=-\frac{y-b}{x-a}.$$

Again, $\partial_y^2\{(x-a)^2+(y-b)^2-c^2\}=0$,

$$\text{or,}\quad \partial_y\{(x-a).\partial_y x+y-b\}=0;$$

$$\therefore\ (x-a).\partial_y^2 x+(\partial_y x)^2+1=0;$$

$$\therefore\ \partial_y^2 x=-\frac{1+\dfrac{(y-b)^2}{(x-a)^2}}{x-a}$$

$$=-\frac{c^2}{(x-a)^3}.$$

Ex. 3. $x^3+y^3-3axy=0$. Required $\partial_x^2 y$.

$$\partial_x(x^3+y^3-3axy)=0;$$

$$\therefore\ 3x^2+3y^2.\partial_x y-3ax.\partial_x y-3ay=0,$$

$$\text{or,}\quad x^2-ay+(y^2-ax).\partial_x y=0.$$

Again, $2x-a.\partial_x y+(y^2-ax).\partial_x^2 y+2y.(\partial_x y)^2-a.\partial_x y=0$;

$$\therefore\ (y^2-ax).\partial_x^2 y=2a.\partial_x y-2y.(\partial_x y)^2-2x$$

$$=2\left\{a.\frac{ay-x^2}{y^2-ax}-y.\left(\frac{ay-x^2}{y^2-ax}\right)^2-x\right\}$$

$$=-\frac{2a^3xy}{(y^2-ax)^2};$$

$$\therefore\ \partial_x^2 y=-\frac{2a^3xy}{(y^2-ax)^3}.$$

Ex. 4. $u = nt + e.\sin u$. Required $\partial_{nt} u$.

$$\text{Since, } u - nt - e.\sin u = 0,$$

$$\partial_{nt}(u - nt - e.\sin u) = 0,$$

$$\text{or, } \partial_{nt} u - 1 - e.\cos u.\partial_{nt} u = 0;$$

$$\therefore \partial_{nt} u = \frac{1}{1 - e.\cos u}.$$

Obs. *Implicit* functions are frequently convertible into *explicit* functions, and their differentials determinable by the rules given in the previous Sections. Thus in Ex. 4, the equation being put under the form, $nt = u - e.\sin u$, nt becomes an explicit function of u, and

$$\partial_u(nt) = 1 - e.\cos u;$$

$$\therefore \text{by Art. 16, } \partial_{nt} u = \frac{1}{1 - e.\cos u}, \text{ as before.}$$

SECTION V.

THE DIFFERENTIATION OF CERTAIN GEOMETRICAL AND PHYSICAL FUNCTIONS OF ONE VARIABLE.

44. MANY quantities occur in Geometry and Physics which, from the definitions given of them, or the ideas attached to their names, are evidently "functions" of each other, although the exact relation which subsists between them may be unknown or arbitrary. Thus, in a plane curve described according to some law, whatever that law may be, the abscissa, ordinate, radius vector, angle between the radius vector and prime radius, arc, area, direction of the tangent, &c., corresponding to the same point, are evidently "functions" of each other; since if a value be given to any one of these quantities, the corresponding point in the curve, and all the others of such quantities, become determinate. And similarly, if a body acted on by a continuously variable force describe a given path, whatever that path may be, the time of its motion from a given epoch, its velocity, the intensity of the force acting upon it, &c., corresponding to any point of its path, are "functions" of each other.

45. Although, so long as the relations between the above quantities are arbitrary or indeterminate, it is impossible to obtain the differential of one of them with regard to another, *in terms of such other*, it is in many cases possible to establish an equation between such differential and some other function, by means of which, when the relation is given, the differential may be found in terms of the quantity with regard to which it is taken.

The following axioms and propositions are of general application in establishing such equations.

46. Axiom I. If $\phi\,\overline{x+h}$ increase or decrease continuously for the successive values of h between 0 and some assumed definite quantity, the difference between $\phi\,\overline{x+h}$ and ϕx can, by diminishing h, be made to decrease continuously and become less than any assignable quantity.

47. Axiom II. If the difference between $\frac{\Delta_x u}{h}$ and any given function of x, as ϕx, can, by diminishing h, be made to decrease continuously and become less than any assignable quantity, $\frac{\Delta_x u}{h} = \phi x$, when 0 is substituted for h.

48. *If* $\frac{\Delta_x u}{h}$ *is equal to* ϕx, *when* 0 *is substituted for* h, ϕx *is, or is equal to, the differential of* u *with regard to* x, *or* $\partial_x u$.

For since $\frac{\Delta_x u}{h}$ is equal to ϕx when 0 is substituted for h; if it be expressed in terms of x and h, the corresponding function will be equal to ϕx, when 0 is substituted for h, and must therefore be capable of being reduced to such a form, that when 0 is substituted for h it becomes either the function ϕx, or some other function equal to it. Hence (Art. 6.) ϕx is or is equal to the differential of u with regard to x, or $\partial_x u$.

49. *If* $\Delta_x u$ *be between* $h.\phi x$ *and* $h.\phi\,\overline{x+h}$ *for all values of* h *between* 0 *and some assumed definite quantity, and* $\phi\,\overline{x+h}$ *increase or decrease continuously between those limits of* h, $\partial_x u = \phi x$.

For it follows from the hypothesis, that

$$\Delta_x u \sim h.\phi x = \text{a quantity not} > h.\{\phi\,\overline{x+h} \sim \phi x\},$$

and $\therefore$ $\frac{\Delta_x u}{h} \sim \phi x = \text{a quantity not} > \{\phi\,\overline{x+h} \sim \phi x\}$,

for all values of h between 0 and the assumed quantity; and by hypothesis $\phi\,\overline{x+h}$ increases or decreases continuously between those limits of h;

Hence, by diminishing h, (Art. 46.) $\phi\, \overline{x+h} \sim \phi x$, and therefore $\frac{\Delta_x u}{h} \sim \phi x$, may be made to decrease continuously and become less than any assignable quantity.

$\therefore$ (Art. 47.) $\frac{\Delta_x u}{h} = \phi x$, when 0 is substituted for h,

and $\therefore$ (Art. 48.) $\partial_x u = \phi x$.

We now proceed to the investigation of some of the equations above referred to for determining the differentials of certain geometrical and physical functions.

GEOMETRICAL FUNCTIONS.

Obs. The propositions which follow respecting curves apply to those only which are described according to laws rendering the abscissæ, ordinates, &c., "functions" of each other.

CURVES REFERRED TO TWO RECTANGULAR CO-ORDINATES.

50. *In a plane curve referred to two rectangular co-ordinates, to find the differential of the* ORDINATE *with regard to the abscissa in terms of the angle which the tangent to the curve at the corresponding point makes with the axis of abscissæ.*

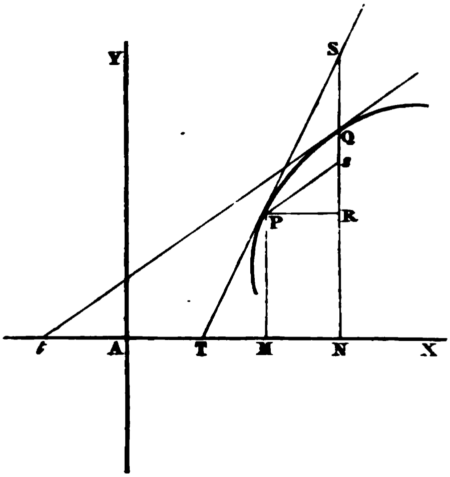
Let AX, AY be the axes of co-ordinates; P, Q points in the curve between which the ordinate, and angle which the tangent to the curve makes with the axis of x, increase or decrease continuously;

$AM\,(x)$, and $MP\,(y)$, the co-ordinates of P;

$AN\,(x+h)$, and NQ, the co-ordinates of Q;

PT, Qt, the tangents to the curve at P and Q intersecting the axis of x in T and t.

Let TP produced meet NQ, produced if necessary, in S;



Draw PR parallel to AX meeting NQ produced if necessary in R;
and Ps .... Qt .... s.

Then, since the magnitude of the angle which the tangent to the curve makes with the axis of x depends upon the abscissa x according to some law, the tangent of that angle must be a "function" of x; and if this function be represented by ϕx,

$$\tan PTX = \phi x,$$

$$\text{and } \tan QtX = \phi\,\overline{x+h}.$$

$$\text{Also,}\quad SR = PR.\tan SPR = MN.\tan PTX,$$

$$\therefore\ SR \doteq h.\phi x$$

$$\text{and similarly,}\quad sR = h.\phi\,\overline{x+h}.$$

But since, by the hypothesis, the ordinate, and angle which the tangent to the curve makes with the axis of x, increase or decrease continuously between the points P and Q, it is evident that the magnitude of QR is between SR, and sR for all points in the curve between those limits; therefore, since QR is $\Delta_x y$,

$$\Delta_x y \text{ is between } h.\phi x \text{ and } h.\phi\,\overline{x+h},$$

for all values of h between 0 and MN.

And, since the angle which the tangent to the curve makes with the axis of x increases or decreases continuously between the points P and Q, and consequently between the above limits of h, the goniometrical tangent of that angle, or $\phi\,\overline{x+h}$, also increases or decreases continuously between those limits.

Hence, (Art. 49.)

$$\begin{aligned} \partial_x y &= \phi x \\ &= \tan PTX. \end{aligned}$$

Obs. As the equations to curves referred to rectangular co-ordinates are generally given immediately between x and y, the above equation can rarely be necessary for determining $\partial_x y$. It will, however, serve for determining the direction of the tangent to a given curve at any point—the curve being given by its equation referred to rectangular co-ordinates—and will also be found useful in determining the differentials of some other geometrical functions.

Note. By following the construction described in the proposition three other figures may be drawn, corresponding to the case when the angle PTX *increases* continuously from P to Q, and to the cases when the ordinates *decrease* continuously from P to Q.

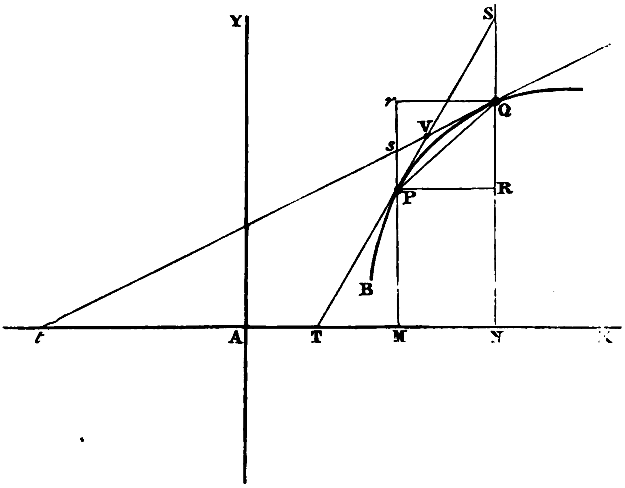
51. *In a plane curve referred to two rectangular co-ordinates, to find the differential of the* ARC *with regard to the abscissa in terms of the differential of the ordinate with regard to the abscissa.*

Let AX, AY, be the axes of co-ordinates, P, Q, points in the curve between which the curve is wholly concave or convex to the axis of x, and the angle which the tangent to the curve makes with the axis of x increases or decreases continuously.

$AM\,(x)$, $MP\,(y)$, the co-ordinates of P,

$AN\,(x+h)$, NQ, the co-ordinates of Q;

TPVS, *tsVQ*, tangents to the curve at *P* and *Q* intersecting each other in *V*, meeting the axis of x in *T*, *t*, and



meeting or intersecting the ordinates *NQ*, *MP*, produced if necessary, in *S*, *s*.

Draw *PR*, *Qr* parallel to *AX* meeting *NQ*, *MP* produced if necessary, in *R*, *r*. Join *P* and *Q*; and let the arc *BP*, measured from any assumed point *B*, be represented by s.

Let the secant of the angle which the tangent to the curve makes with the axis of x, and which is evidently a function of x, be represented by ϕx, so that $\sec PTX = \phi x$, $\sec QtX = \phi\,\overline{x+h}$.

Then, since the arc *PQ* is wholly concave or convex to the axis of x, it is evident from the construction that the curve *PQ* and the rectilinear figure *PVQ*, terminated at the same points, have their convexities in the same direction, do not cut each other, and cannot respectively be intersected by a straight line in more points than two;

∴ by Lemma 1, page 28, the arc *PQ* is in magnitude between the chord *PQ* and $PV + VQ$ the perimeter of the rectilinear figure *PVQ*.

Again, if each of the equal angles SQV, VsP be $>$ a right angle,

$$VS \text{ is } > VQ;$$

$$\therefore\ PS\,(= PV + VS) \text{ is } > PV + VQ,$$

and $\therefore$ $$> \text{arc } PQ.$$

Also, $$Qs \text{ is } < \text{chord } PQ,$$

and $\therefore$ $$< \text{arc } PQ.$$

Hence, when each of the equal angles SQV, VsP is $>$ a right angle, the arc PQ is in magnitude between PS and Qs. And the same may in like manner be proved, when each of the angles SQV, VsP is $=$ or $<$ a right angle.

Therefore, since the angle, which the tangent to the curve makes with the axis of x, increases or decreases continuously between the points P and Q, the arc measured from P to *any* point between P and Q is in magnitude between the parts of the tangents at those points intercepted between the corresponding ordinates, produced if necessary.

$$\begin{aligned} \text{Now,}\quad PS &= PR.\sec SPR \\ &= MN.\sec PTX \\ &= h.\phi x. \\ \text{Also,}\quad Qs &= Qr.\sec sQr \\ &= MN.\sec QtX \\ &= h.\phi\,\overline{x+h}. \end{aligned}$$

$$\text{And,}\quad PQ \text{ is } \Delta_x s.$$

Hence, $\Delta_x s$ is between $h.\phi x$ and $h.\phi\,\overline{x+h}$, for all arcs measured from P to any point between P and Q, and consequently for all values of h between 0 and MN. And since the angle which the tangent to the curve makes with the axis of x increases or decreases continuously between the points P and Q, $\phi\,\overline{x+h}$, the secant of that angle, increases or decreases continuously between the above limits of h;

$$\therefore\ (\text{Art. } 49.)\qquad \partial_x s = \phi x.$$

$$\text{But } \phi x = \sec PTX = \sqrt{1 + (\tan PTX)^2}$$

$$= \sqrt{1 + (\partial_x y)^2};$$

$$\therefore\ \partial_x s = \sqrt{1 + (\partial_x y)^2};$$

from which, when the equation to the curve is given between x and y, $\partial_x s$ may be found in terms of x and y, or of either of them in case the given equation between x and y can be solved.

Cor. Since $\partial_x s = \dfrac{\partial s}{\partial x}$

$$\partial s = \partial x . \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}$$

$$= \sqrt{(\partial x)^2 + (\partial y)^2}.$$

Note. The Note appended to Art. 50. is applicable to this proposition also.

52. *In a plane curve referred to rectangular co-ordinates, to find the differential with regard to the abscissa of the* AREA *bounded by the curve, the axis of abcissæ, the ordinate, and any other assumed ordinate.*

Let AX, AY, be the axes of co-ordinates.

P and Q points in the curve between which the ordinate increases or decreases continuously.

AM (x), MP (y),

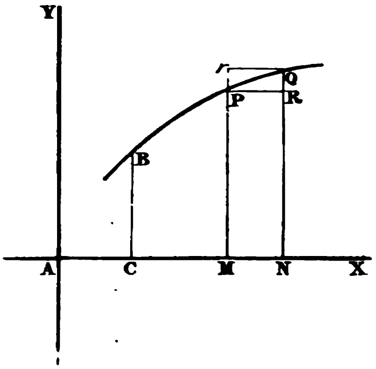
co-ordinates of P,

AN $(x + h)$, NQ,

co-ordinates of Q,

BC the assumed ordinate,

$BPMC$ (A) the area.



Draw PR, Qr parallel to AX meeting NQ and MP produced if necessary in R, r; and as y is a function of x, let it be represented by ϕx.

Then $MP = \phi x$ and $NQ = \phi\,\overline{x+h}$.

But since by hypothesis the ordinate increases or decreases continuously between the points P and Q, it is evident that

area $PMNQ$ is between rectangles $PMNR$ and $QNMr$,

i. e. .... $MN.MP$, and $MN.NQ$;

and that any other similar area intercepted between MP and the ordinate to *any* point betwixt P and Q will be between the corresponding rectangles.

Therefore, since $PMNQ$ is $\Delta_x A$, $MN = h$, $MP = \phi x$, and $NQ = \phi\overline{x+h}$,

$$\Delta_x A \text{ is between } h.\phi x \text{ and } h.\phi\overline{x+h}$$

for all such areas, and consequently for all values of h between 0 and MN; and since by hypothesis the ordinate, and therefore $\phi\overline{x+h}$, increases or decreases continuously between those limits of h,

$$\therefore \text{ (Art. 49.) } \partial_x A = \phi x$$
$$= y;$$

from which, when the equation to the curve is given between x and y, $\partial_x A$ may be found in terms of x, provided that the given equation can be solved with respect to y.

NOTE. As in Arts. 50 and 51, three other figures may be drawn here also.

If the curve were referred to *oblique* co-ordinates the area $PMNQ$ would lie between the parallelograms $PMNR$ and $QNMr$,

i. e. between $MN.MP.\sin YAX$, and $MN.NQ.\sin YAX$.

Therefore, if $\angle\, YAX = i$,

$$\Delta_x A \text{ is between } h.\phi x.\sin i \text{ and } h.\phi\overline{x+h}.\sin i,$$

for all values of h between 0 and MN;

and reasoning as before,

$$\partial_x A = \phi x.\sin i$$
$$= y.\sin i.$$

CURVES REFERRED TO POLAR CO-ORDINATES.

Def. A curve (see Fig. Art. 53.) is said to be referred to polar co-ordinates when the equation to it is given between the line SP, called the "*radius vector*", (drawn from a given point S, called the "*pole*", to the corresponding point P in the curve,) and the angle ASP which such radius makes with some other assumed radius SA, which latter radius is called the "*prime radius*". The angle ASP is sometimes termed the "*generating angle*".

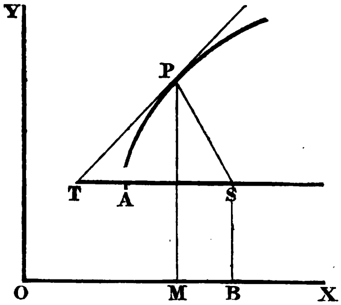
53. *In a plane curve referred to polar co-ordinates, to find the differential of the* RADIUS VECTOR *in terms of such radius and the angle which it makes with the tangent at the corresponding point of the curve.*

Let $\angle ASP(\theta)$, $SP(r)$ be the polar co-ordinates of any point P of the curve, SA being the "prime radius".

And let $OM(x)$, $MP(y)$ be the rectangular co-ordinates of P,

$OB(a)$, $BS(b)$, ... S,

considered as respectively referred to the axes OX, OY, one of which, OX, is parallel to SA.



$$\text{Then}\quad x = a - r.\cos\theta,$$

$$y = b + r.\sin\theta;$$

$$\therefore\ \partial_\theta x = r.\sin\theta - \cos\theta.\partial_\theta r,$$

$$\partial_\theta y = r.\cos\theta + \sin\theta.\partial_\theta r;$$

$$\therefore \quad \partial_x y = \frac{\partial_\theta y}{\partial_\theta x} = \frac{r.\cos\theta + \sin\theta.\partial_\theta r}{r.\sin\theta - \cos\theta.\partial_\theta r}$$

$$= \frac{r.\partial_r\theta + \tan\theta}{r.\partial_r\theta.\tan\theta - 1};$$

$$\therefore \quad \text{(Art. 50.)} \quad \tan STP = \partial_x y = -\tan\{\theta + \tan^{-1}(r.\partial_r\theta)\};$$

$$\therefore \quad STP = 180^0 - \theta - \tan^{-1}(r.\partial_r\theta);$$

$$\therefore \quad \tan^{-1}(r.\partial_r\theta) = 180^0 - \overline{\theta + STP}$$

$$= \angle SPT;$$

$$\therefore \quad r.\partial_r\theta = \tan SPT,$$

$$\text{and} \quad \partial_\theta r = r.\cot SPT.$$

From this equation the angle which the radius vector makes with the tangent to the curve at any point may be determined, when the equation to the curve is given referred to polar co-ordinates.

Cor. 1. If the radius vector be represented by $\frac{1}{u}$, as is sometimes the case in Treatises on Dynamics,

$$\frac{\partial_\theta r}{r} = -\frac{\partial_\theta u}{u};$$

$$\therefore \quad \tan SPT = -\frac{u}{\partial_\theta u}.$$

Cor. 2. If the perpendicular from the pole upon the tangent be represented by p,

$$\partial_\theta r = r.\cot SPT$$

$$= r.\sqrt{(\operatorname{cosec} SPT)^2 - 1}$$

$$= r.\sqrt{\frac{r^2}{p^2} - 1}$$

$$= \frac{r}{p}\sqrt{r^2 - p^2}.$$

54. *In a plane curve referred to polar co-ordinates, to find the differential of the* ARC *with regard to the "generating angle" in terms of the radius vector and its differential with regard to the same angle.*

If the curve be considered as given by an equation between rectangular co-ordinates, the differential of the arc taken with regard to an arbitrary variable is determined from the equation

$$\partial s = \sqrt{(\partial x)^2 + (\partial y)^2}. \quad \text{(Art. 51.)}$$

And, if the same curve be also referred to polar co-ordinates, the prime radius being parallel to the axis of x, and a, b being the rectangular co-ordinates of the pole, as in the preceding Article, x and y are functions of r and θ determinable from the equations

$$x = a - r\cos\theta,$$

$$y = b + r\sin\theta;$$

$$\text{hence, } \partial x = r.\sin\theta.\partial\theta - \cos\theta.\partial r,$$

$$\partial y = r.\cos\theta.\partial\theta + \sin\theta.\partial r;$$

$$\therefore\ (\partial x)^2 + (\partial y)^2 = r^2.\{(\sin\theta)^2 + (\cos\theta)^2\}.(\partial\theta)^2 + \{(\sin\theta)^2 + (\cos\theta)^2\}.(\partial r)^2.$$

$$= r^2.(\partial\theta)^2 + (\partial r)^2;$$

$$\therefore\ \partial s = \sqrt{r^2.(\partial\theta)^2 + (\partial r)^2}$$

$$\text{and, } \partial_\theta s = \sqrt{r^2 + (\partial_\theta r)^2}.$$

From this equation, when the equation to the curve is given between r and θ, $\partial_\theta s$ may be found in terms of θ, provided that the given equation can be solved with respect to r.

COR. 1. Since $\partial s = \partial_\theta s.\partial\theta$, and $\partial_\theta r = \dfrac{\partial r}{\partial\theta}$,

$$\partial s = \sqrt{r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2}.\partial\theta$$

$$= \sqrt{(r.\partial\theta)^2 + (\partial r)^2}.$$

Cor. 2. If $r = \frac{1}{u}$,

$$\partial_\theta s = \sqrt{\frac{1}{u^2} + \left(\frac{\partial_\theta u}{u^2}\right)^2}$$

$$= \frac{1}{u^2}.\sqrt{u^2 + (\partial_\theta u)^2}.$$

Cor. 3. If the perpendicular from the pole upon the tangent be represented by p,

$$p = r.\sin SPT$$

$$= \frac{r}{\operatorname{cosec} SPT}$$

$$= \frac{r}{\sqrt{1 + (\cot SPT)^2}}$$

$$= \frac{r}{\sqrt{1 + \left(\frac{\partial_\theta r}{r}\right)^2}}$$

$$= \frac{r^2}{\sqrt{r^2 + (\partial_\theta r)^2}}.$$

Or, if $r = \frac{1}{u}$, $p = \frac{1}{\sqrt{u^2 + (\partial_\theta u)^2}}$;

$$\therefore\ \partial_\theta s = \frac{r^2}{p},$$

$$\text{or} \quad = \frac{1}{pu^2}.$$

Also $r^2.\partial\theta = p.\partial s$.

Obs. The equations deduced in the corollaries of this and the preceding Article are of frequent occurrence in Treatises on Dynamics.

55. **Lemma.** *If the angle at the centre of a circle subtended by an arc which is equal to the radius be taken as unity, and θ be the number of such angles in a given angle, the area of the corresponding sector of the circle* $= \frac{\theta}{2}.(radius)^2$.

Divide the given angle θ into angles each equal to α, then the number of such angles is $\frac{\theta}{\alpha}$. Join each two adjacent points in which the lines forming these angles meet the circumference, and also draw tangents through the same points, so as to form an inscribed and a circumscribed rectilinear figure: then it is evident that the area of the sector is always between the areas of these two figures.

Now, if a be the radius of the circle, the inscribed figure will consist of $\frac{\theta}{\alpha}$ triangles, each having two sides $= a$ and including an angle α; and the circumscribed figure will be equal to $\frac{\theta}{\alpha}$ triangles, each having two sides $= a.\sec\frac{\alpha}{2}$ and including an angle $= \alpha$;

$$\therefore \text{ area of inscribed figure} = \frac{\theta}{\alpha}.\frac{1}{2}.a^2.\sin\alpha$$

$$= \frac{\theta}{2}.a^2.\frac{\sin\alpha}{\alpha}.$$

$$\ldots\ldots \text{circumscribed} \ldots\ldots = \frac{\theta}{\alpha}.\frac{1}{2}.\left(a.\sec\frac{\alpha}{2}\right)^2.\sin\alpha$$

$$= \frac{\theta}{2}.a^2.\frac{\sin\alpha}{\alpha}.\left(\sec\frac{\alpha}{2}\right)^2.$$

Hence, sector $\sim \frac{\sin\alpha}{\alpha}.\frac{\theta}{2}.a^2$ is $< \frac{\sin\alpha}{\alpha}.\frac{\theta}{2}.a^2.\left\{\left(\sec\frac{\alpha}{2}\right)^2 \sim 1\right\}$, or, sector $\sim$ (1 + a quantity not $> \tan\frac{a}{2}$ and not neg.)$^{-1}.\frac{\theta}{2}.a^2$

is $<$ (1 + a quantity not $> \tan\frac{a}{2}$ and not neg.)$^{-1}.\frac{\theta}{2}.a^2$

$$\times\left\{\left(\sec\frac{a}{2}\right)^2 \sim 1\right\},$$

for all values of a between the limits 0 and θ. And since, when a is diminished, $\sec\frac{a}{2}$ approaches continuously to 1, it is evident that the above difference can, by diminishing a, be made to decrease continuously and become less than any assignable quantity;

$\therefore$ sector = (1 + a quantity not $> \tan\frac{a}{2}$ and not neg.)$^{-1}.\frac{\theta}{2}a^2$,

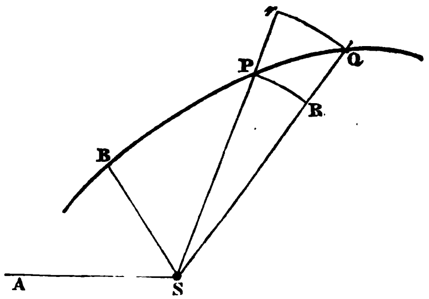
when 0 is substituted for h;

$$\text{and } \therefore \text{ sector} = \frac{\theta}{2}.a^2.$$

Cor. If 2π represent the whole circumference of a circle,

$$\text{area of circle} = \pi.(\text{radius})^2.$$

56. *In a plane curve referred to polar co-ordinates, to find the differential with regard to the generating angle of the* AREA *bounded by the curve, the radius vector, and any other assumed radius.*



Let SA be the prime radius, P and Q points in the curve between which the radius vector increases or decreases continuously.

$ASP(\theta)$, $SP(r)$

the co-ordinates of P,

$ASQ(\theta+h)$, SQ

the co-ordinates of Q;

SB the assumed radius,

$BSP(V)$ the area;

PR, *Qr* circular arcs described with center *S*, meeting *SQ* and *SP*, produced if necessary, in *R* and *r*.

And, since r^2 is a function of θ, let $\frac{r^2}{2}$ be represented by $\phi\theta$;

$$\text{then, } \frac{SP^2}{2} = \phi\theta, \quad \frac{SQ^2}{2} = \phi\,\overline{\theta+h};$$

But, since by hypothesis the radius vector increases or decreases continuously between the points *P* and *Q*, it is evident that the area *PSQ* is between the circular sectors *PSR* and *QSr*; and that any other similar area intercepted between the radius *SP* and the radius corresponding to any point in the curve between *P* and *Q* will be between the corresponding circular sectors.

Therefore, since *PSQ* is $\Delta_\theta V$, $\angle\, PSQ = h$, and the circular sectors *PRS*, *QSr* are equal to $\frac{h}{2}.(SP)^2$, and $\frac{h}{2}.(SQ)^2$ respectively,

$\Delta_\theta V$ is between $h.\phi\theta$ and $h.\phi\,\overline{\theta+h}$, for all such areas, and consequently for all values of h between 0 and *PSQ*.

And since, by hypothesis, the radius vector, and consequently $\phi\,\overline{\theta+h}$, increases or decreases continuously between those limits of h,

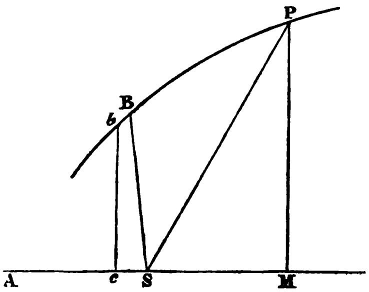
$$\partial_\theta V = \phi\theta$$
$$= \frac{r^2}{2}.$$

From this equation, when the equation to the curve is given between r and θ, $\partial_\theta V$ may be found in terms of θ, provided that the equation can be solved with respect to r.

Cor. 1. Hence $\partial V = \frac{r^2}{2}.\partial\theta$, and if $u = \frac{1}{r}$,

$$\partial_\theta V = \frac{1}{2u^2}$$

$$\text{and } \partial V = \frac{1}{2u^2}.\partial\theta.$$



Cor. 2. If the curve be also given by an equation between rectangular co-ordinates, the origin of co-ordinates being S, and AS produced the axis of x; and if SB be the assumed radius bounding the area $BSP(V)$, and bc the assumed ordinate bounding the area $bcMP(A)$; it is evident that

$$V = A - PSM - bcSB,$$

which last area is a constant quantity (C);

$$\therefore\ V = A - \tfrac{1}{2}xy - C;$$

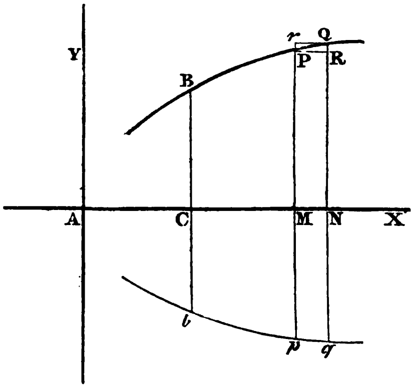
$$\therefore\ \partial V = \partial A - \tfrac{1}{2}(y.\partial x + x.\partial y),$$

$$\text{(Art. 52.)} \qquad = y.\partial x - \tfrac{1}{2}(y.\partial x + x.\partial y)$$

$$= \tfrac{1}{2}(y.\partial x - x.\partial y).$$

SURFACES AND SOLIDS CONSIDERED AS FUNCTIONS OF ONE VARIABLE.

57. *In a surface formed by the revolution of a plane curve referred to rectangular co-ordinates round the axis of abscissæ, to find the differential of the* surface, *bounded by the circles whose radii are respectively the ordinate and any other assumed ordinate, with regard to the arc of the generating curve.*



Let AX, AY be the axes of co-ordinates; P, Q, points in the curve between which the ordinate increases or decreases continuously; BC the assumed ordinate; BP (s), MP (y), the arc and ordinate corresponding to P; BQ $(s+h)$, NQ, the arc and ordinate corresponding to Q; S the surface formed by the revolution of BP round the axis of x.

Then $\Delta_{,}S$ is the surface formed by the revolution of PQ.

Now, since the ordinate increases or decreases continuously between P and Q, it is evident that $\Delta_{,}S$ is between the cylindrical surfaces which would be formed by the revolution of a straight line $= PQ$ at the distances MP, NQ respectively from the axis,

i. e. $\Delta_{,}S$ is between $2\pi.MP.PQ$, and $2\pi.NQ.PQ$;

and it is also evident that the surface intercepted between the circle described by MP and that described by the ordinate to any other point in the curve between P and Q will be between the corresponding cylindrical surfaces.

But MP is a function of the arc BP or s;

$\therefore$ if $2\pi.MP$ be represented by ϕs,

$2\pi.NQ$ will be represented by $\phi\,\overline{s+h}$.

$\therefore$ $\Delta_{,}S$ is between $h.\phi s$ and $h.\phi\,\overline{s+h}$,

for all such areas, and therefore for all values of h between 0 and $\overline{PQ}$; and since by hypothesis the ordinate, and therefore $\phi\,\overline{s+h}$, increases or decreases continuously between those limits of h,

$$\text{(Art. 49)},\quad \partial_{,}S = \phi s$$
$$= 2\pi y.$$
$$\text{And } \partial S = 2\pi y.\partial s.$$

58. *In a solid formed by the revolution of a plane curve referred to rectangular co-ordinates round the axis of abscissæ, to find the differential with regard to the abscissa of the* VOLUME *bounded by the circles whose radii are respectively the ordinate and any other assumed ordinate.*

Let AX, AY be the axes of co-ordinates, (fig. Art. 57.) P, Q points in the curve between which the ordinate increases or decreases continuously;

BC the assumed ordinate;

$AM\,(x)$, $MP\,(y)$, co-ordinates of P,

$AN\,(x+h)$, NQ, co-ordinates of Q;

V the volume intercepted between the circles described by CB and MP.

Then $\Delta_x V$ is the volume between the circles described by MP and NQ.

Draw PR, Qr parallel to AX, meeting NQ, MP, produced if necessary, in R and r.

Since the ordinate increases or decreases continuously between P and Q, it is evident that $\Delta_x V$ is between the cylinders which would be formed by the revolution of the rectangles $MPRN$, and $Mr\,QN$, round the axis AX,

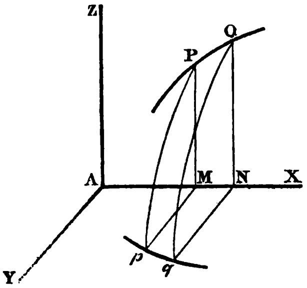
i. e. between $\pi.MP^2.MN$, and $\pi.NQ^2.MN$;

and assuming $\pi.MP^2 = \phi x$, and $\therefore$ $\pi.NQ^2 = \phi\,\overline{x+h}$, and proceeding as in the preceding proposition, it may be shewn that

$$\begin{aligned} \partial_x V &= \phi x \\ &= \pi y^2. \end{aligned}$$

$$\text{And } \partial V = \pi y^2.\partial x.$$

59. *To find the differential of the* VOLUME *of a body (which is bounded by a surface generated according to some law) with regard to the distance from a given plane of a section of the body made by a plane parallel to the given plane, in terms of the area of such section.*



Let YAZ be the given plane, Pp, Qq sections of the body made by planes parallel to YAZ, between which the areas of such sections increase or decrease continuously.

$AM\,(x)$, $AN\,(x+h)$ the distances of the planes Pp, Qq from the given plane.

A the area of the section Pp;

V the volume of the body cut off by the plane Pp;

$V + \Delta_x V$ .... Qq;

And as the area A is a function of x, let it be represented by ϕx, then the area corresponding to Q will be represented by $\phi\,\overline{x+h}$.

Now, since by the hypothesis the areas of the sections parallel to the given plane increase or decrease continuously between P and Q, it is evident that the volume intercepted between Pp and Qq is in magnitude between the cylinders whose height is h and whose bases are the areas of the sections at P and Q respectively; and that the same will be true if any other point between P and Q be taken for Q.

Hence, $\Delta_x V$ is between $h.\phi x$ and $h.\phi\,\overline{x+h}$ for all values of h between 0 and MN.

And by hypothesis $\phi\,\overline{x+h}$ increases or decreases continuously between the same limits of h,

$$\therefore \text{ (Art. 49.) } \quad \partial_x V = \phi x$$
$$= A.$$

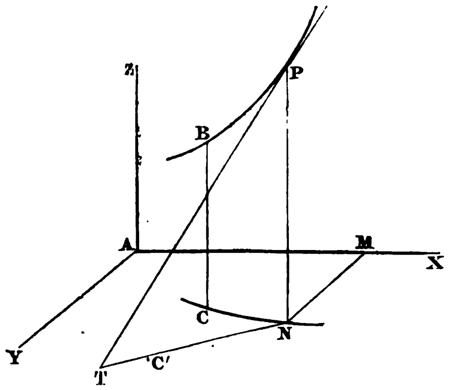
Hence also, $\partial V = A.\partial x$.

Cor. If the body be formed by the revolution of a plane curve about AX, and if $PM = y$, $A = \pi y^2$,

$$\therefore \ \partial V = \pi y^2.\partial x; \text{ as in Art. 58.}$$

CURVES REFERRED TO THREE RECTANGULAR CO-ORDINATES.

60. *In a curve either plane or of double curvature referred to three rectangular co-ordinates, to find the angles which the tangent to the curve at any point makes with the co-ordinate planes in terms of the differentials of the co-ordinates.*



Let AX, AY, AZ be the axes of co-ordinates, BP the curve, $AM\,(x)$, $MN\,(y)$, $NP\,(z)$ co-ordinates of P; PT

the tangent at *P* meeting the plane *XAY*, or the plane of *xy*, in *T*.

Join *TN*; then *PTN* is the angle which the tangent makes with the plane of *xy*.

Let *CN* be the projection of the curve formed by drawing perpendiculars from every point between *B* and *P* to the plane of *xy*; then it is evident that the surface formed by these perpendiculars is cylindrical, and may therefore be developed or unrolled. Suppose it to be so unrolled and made to coincide with the plane *PTM*, the line *PN* being supposed fixed.

Then the curve *PB* will become a plane curve lying in the plane *PTN*; and *NC* will become a straight line in that plane coinciding with *NT*; and if *C* coincides with *C'*, *C'N*, *NP* may be considered as co-ordinates of the curve *BP* when reduced to a plane curve in the manner above described.

If therefore $CN = \sigma$, $C'N = \sigma$;

$$\text{and} \quad \tan PTN = \frac{\partial z}{\partial \sigma}. \quad \text{(Art. 50.)}$$

But σ is the arc of the plane curve *CN* whose co-ordinates are x and y,

$$\therefore \partial\sigma = \sqrt{(\partial x)^2 + (\partial y)^2};$$

$$\therefore \tan PTN = \frac{\partial z}{\sqrt{(\partial x)^2 + (\partial y)^2}};$$

and the angles which the tangent to the curve makes with the other co-ordinate planes may be found from similar equations.

Cor. If θ_x, θ_y, θ_z, be the angles which the tangent to the curve at any point makes with lines drawn through that point parallel to the axes of co-ordinates x, y, z respectively,

$$\text{since } \angle PTN = 90^0 - \theta_z;$$

$$\therefore \ \cot \theta_z = \frac{\partial z}{\sqrt{(\partial x)^2 + (\partial y)^2}} = \frac{1}{\sqrt{(\partial_z x)^2 + (\partial_z y)^2}}$$

$$\text{or,} \ \tan \theta_z = \sqrt{(\partial_z x)^2 + (\partial_z y)^2}.$$

$$\text{Similarly,} \ \tan \theta_y = \sqrt{(\partial_y x)^2 + (\partial_y z)^2}$$

$$\tan \theta_x = \sqrt{(\partial_x y)^2 + (\partial_x z)^2}.$$

61. *In a curve either plane or of double curvature referred to three rectangular co-ordinates, to find the differential of the* ARC *in terms of the differentials of the co-ordinates.*

Let AX, AY, AZ be the axes of co-ordinates, (fig. Art. 60.) BP the curve, B being some assumed point in it; CN the projection of BP on the plane of xy, formed by drawing perpendiculars from every point in BP to the plane of xy;

$AM\,(x)$, $MN\,(y)$, and $NP\,(z)$, the co-ordinates of P;

and let the arc BP measured from the assumed point B be represented by s, and the arc CN measured from the corresponding point C be represented by σ.

Then, in the same manner as in the preceding proposition, BP may be considered as developed into a plane curve, the co-ordinates of which are CN, NP, or σ and z;

$$\therefore \ \text{(Art. 51.)} \quad \partial s = \sqrt{(\partial \sigma)^2 + (\partial z)^2}.$$

And since CN or σ is the arc of a plane curve, whose co-ordinates are x and y,

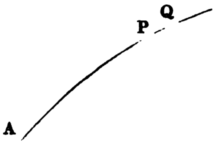
$$(\partial \sigma)^2 = (\partial x)^2 + (\partial y)^2;$$

$$\therefore \ \partial s = \sqrt{(\partial x)^2 + (\partial y)^2 + (\partial z)^2}.$$

PHYSICAL FUNCTIONS.

62. *If a body considered as a point move with a continuously varying velocity, to find the differential of the* SPACE *described with regard to the time of its motion from a given epoch in terms of the velocity at the end of such time.*

Let APQ be the path of the body; P, Q points between which the velocity increases or decreases continuously.



$AP\,(s)$ the space described by the body in the time t from the given epoch,

$AQ\,(s+\Delta_t s)$ the space described by the body in the time $\overline{t+h}$ from the given epoch;

v the velocity at the point P:—

and, as v is a function of t, let it be represented by ϕt; then the velocity at Q will be represented by $\phi\,\overline{t+h}$.

Since by hypothesis the velocity increases or decreases continuously between P and Q, it is evident that the space PQ is in magnitude between the spaces which would be described by the body moving uniformly during the time h with the velocities at the points P and Q respectively, i. e. with the velocities ϕt and $\phi\,\overline{t+h}$; and that the same will be true if any other point between P and Q be taken instead of Q.

Hence, $\Delta_t s$ is between $h.\phi t$ and $h.\phi\,\overline{t+h}$, (Whewell's Mechanics 5th Ed. Art. 104.) for all values of h between 0 and its value at the point Q; and by hypothesis $\phi\,\overline{t+h}$ increases or decreases continuously between the same limits of h,

$$\therefore \text{ (Art. 49.) } \quad \partial_t s = \phi t$$

$$= v.$$

Hence also, $\partial s = v.\partial t$.

63. *If a body considered as a point be acted on by a continuously varying force in the direction of its motion, to find the differential of the* VELOCITY *of the body with regard to the time of its motion from a given epoch in terms of the force at the end of such time.*

Let *APQ* (Fig. Art. 62.) be the path of the body;

P, *Q* points between which the force increases or decreases continuously:

v the velocity at *P* corresponding to the time t from the given epoch;

$v+\Delta_t v$ the velocity at *Q* corresponding to the time $\overline{t+h}$ from the given epoch;

f the force at *P*; and, as f is a function of t, let it be represented by ϕt, then the force at *Q* will be represented by $\phi\,\overline{t+h}$.

Since by hypothesis the force increases or decreases continuously between *P* and *Q*, it is evident that the velocity generated in the time h of the body's motion from *P* to *Q* is between the velocities which would be generated in the same time by the forces at the points *P* and *Q* respectively, i. e. by ϕt and $\phi\,\overline{t+h}$, acting uniformly; and that the same will be true if any other point between *P* and *Q* be taken instead of *Q*.

Hence, $\Delta_t v$ is between $h.\phi t$ and $h.\phi\,\overline{t+h}$, (Whewell's Mechanics, Art. 114.) for all values of h between 0 and its value at the point *Q*; and by hypothesis $\phi\,\overline{t+h}$ increases or decreases continuously between the same limits of h,

$$\therefore \text{ (Art. 49.) } \quad \partial_t v = \phi t$$
$$= f.$$

$$\text{Hence also, } \partial v = f.\partial t.$$

Cor. If the force vary continuously it is evident that the velocity will also vary continuously; and therefore if s be the space described by the body in the time t, when acted upon by a continuously varying force, and v the velocity at the end of the same time,

$$\partial_t s = v, \quad \text{(Art. 62.)}$$

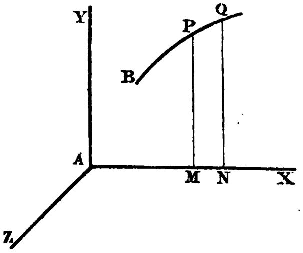
$$\text{and } \partial_t v = f,$$

$$\therefore \partial_t^2 s = f.$$

$$\begin{aligned} \text{Also, } f &= \partial_t v \\ &= \partial_s v . \partial_t s \\ &= v . \partial_s v. \end{aligned}$$

64. *In a continuous body of uniform density generated according to some law, to find the differential of the sum of the products of every particle multiplied by its distance from a given plane with regard to the mass of the body terminated by any plane parallel to the given plane.*

Let the plane YAZ at right angles to AX be the given plane; BPQ the section of the body made by the plane YAX; $AM(x)$, AN, the distances from the given plane of the planes parallel to it through P and Q;



m the mass of the body terminated by the plane through P,
$m+h$ .... Q;

M the sum of the products corresponding to the mass m,
$M + \Delta_x M$ .... $m+h$;

and, as x is a function of m, let it be represented by ϕm; then AN will be represented by $\phi\,\overline{m+h}$.

Now, it is evident that the sum of the products of every particle of the body intercepted between the planes through P and Q multiplied by the distance of such particle from the given plane will be between the products of the whole mass intercepted between those planes multiplied by the distances

AM and *AN* respectively; and that the same will be true if any point between *P* and *Q* be taken instead of *Q*.

Hence, $\Delta_m M$ is between $h.\phi m$ and $h.\phi\,\overline{m+h}$, for all values of h between 0 and its value corresponding to the point *Q*; and it is evident that the distance $\phi\,\overline{m+h}$ increases continuously between the same limits of h,

$$\therefore \text{ (Art. 49.) } \partial_m M = \phi m$$

$$= x.$$

$$\text{And } \partial M = x.\partial m.^{*}$$

Cor. 1. If the body be of uniform thickness (t) with its plane faces parallel to the plane *YAX*, and if *A* be the area of one of the faces, and therefore the area of the section of the body by the plane *YAX*, and ρ the density of the body,

$$m = \rho.t.A;$$

$$\therefore \partial m = \rho t.\partial A$$

$$= \rho t.y.\partial x \text{ (Art. 52.)}$$

$$\text{and } \partial M = \rho t.xy.\partial x.$$

Cor. 2. If the body be a shell of inconsiderable thickness generated by the revolution of a physical line about the axis *AX*, *S* its surface, and s the arc *BP* of the section *BPQ*,

$$m = \rho S;$$

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[*] If the body be supposed to be acted on by the force of gravity in direction parallel to *YA*, *M* is the "*moment*" of the body round the horizontal axis *AZ*; and since, when a body is acted upon by gravity its moment round an horizontal axis is the same as if the whole mass were collected at its "*centre of gravity*," it follows that,

mass of the body × distance of centre of gravity from *YAZ* = whole moment, and ∴ if $\bar{x}$ be this distance,

$$\bar{x} = \frac{\text{whole moment}}{\text{mass of body}}$$

$$= \frac{M}{m}$$

$$= \frac{\int(x.\partial m)}{m},$$

the symbol $\int$ indicating the operation by which, according to the principles of the "Integral Calculus," *M* is obtained from ∂M.

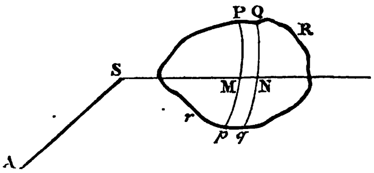
$$\therefore\ \partial m = \rho.\partial S$$
$$= \rho.2\pi y.\partial s,\quad \text{(Art. 57.)}$$
$$= \rho.2\pi y.\sqrt{(\partial x)^2 + (\partial y)^2},$$
$$\text{and } \partial M = \rho.2\pi xy\sqrt{(\partial x)^2 + (\partial y)^2}.$$

COR. 3. If the body be a solid of revolution whose axis is AX, and volume V,

$$m = \rho V;$$
$$\therefore\ \partial m = \rho.\partial V$$
$$= \rho.\pi y^2.\partial x,\quad \text{(Art. 58.)}$$
$$\text{and } \partial M = \rho.\pi xy^2.\partial x.$$

From these several equations, when the relation between x and y is given, the value of $\partial_x M$ may be found in terms of x; and the same method may be extended to any other body of given density when the differential of its volume is known.

65. *In a continuous body of uniform density, to find the differential of the sum of the products of every particle multiplied by the square of its distance from a given line with regard to the mass of the body terminated by any cylindrical surface which has the given line for its axis.*



Let SA be the given line; $PQRqpr$ the section of the body made by a plane through S at right angles to SA; PMp, QNq, sections of the plane $PQRr$ by cylinders which have SA for their common axis, and between which the mass of the body increases or decreases continuously; $SM\,(r)$, SN, the radii of these cylinders;

m the mass of the body terminated by the cylinder through P,

$m + h$ .... Q;

μ the sum of the products corresponding to the mass m,

$\mu + \Delta_m\mu$ .... $m + h$;

and, as r^2 is a function of m, let it be represented by ϕm; then SN^2 will be represented by $\phi\,\overline{m+h}$.

Now, it is evident that the sum of the products of every particle of the body intercepted between the cylinders through P and Q multiplied by the square of the distance of such particle from the given line will be between the products of the whole mass intercepted between those cylinders multiplied by the squares of the radii SM and SN respectively; and that the same will be true if any point between P and Q be taken instead of Q.

Hence $\Delta_m \mu$ is between ϕm and $\phi\,\overline{m+h}$, for all values of h between 0 and its value corresponding to the point Q; and it is evident that SN^2, or $\phi\,\overline{m+h}$ increases continuously between the same limits of h,

$$\therefore \text{ (Art. 49.) } \partial_m \mu = \phi m$$

$$= r^2.$$

$$\text{And } \partial\mu = r^2.\partial m.\text{*}$$

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* The quantity μ, determined from the equation $\partial\mu = r^2.\partial m$ by the rules of the Integral Calculus, is called the "*moment of Inertia*" of the body about the given line. (See Whewell's Dynamics, Part II. Art. 206.)

SECTION VI.

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES.

66. If a quantity be a function of several others which are independent of each other, or are treated as such, its variation may arise from the variation of one only of such quantities, or of all or any number of them considered as arbitrary functions of an undetermined variable; such latter functions being independent, when the several quantities are independent of each other.

Hence the differentiation of functions of several variables may be considered under the two heads of "*partial*" differentiation and "*total*" differentiation.

Thus, if $x^2.\sin y$ be considered as a function of x and y, its "partial differential", obtained by supposing x only to vary, is $2x.\sin y$ (Art. 6), y being treated as a "constant" quantity:—its "partial differential", obtained by supposing y only to vary, is $x^2.\cos y$ (Art. 34); and its "total differential", obtained by supposing both x and y to be functions of an undetermined variable and to vary, is $2x.\sin y.\partial x + x^2.\cos y.\partial y$ (Art. 23); in which last expression, if x and y be independent of each other, both ∂x and ∂y are arbitrary and independent.

Again, if $x.\sin x$ be considered as a function of x and $\sin x$, the "partial" and "total" differentials are respectively $\sin x$, x, and $\sin x.\partial x + x.\partial \sin x$, in which last expression ∂x and $\partial \sin x$ are arbitrary but not independent of each other, since they are connected by the equation $\partial \sin x = \cos x.\partial x$.

It is evident from the mode in which the "partial" and "total" differentials are obtained that the results will be the same whether the variables are in fact independent of each other, or are merely treated as such, and that the consideration of the relations (if any) which subsist between them need not be introduced until the process of differentiation has been completed.

Hence any general propositions proved for differentials of functions of *independent* variables will hold for differentials of functions of *dependent* variables, unless the proof of any such proposition depends upon the *fact* that the variables are independent.

Hence also, the propositions proved in the preceding Sections respecting differentials of functions of one variable will apply to "*partial*" differentials of functions of *several* variables, or to "*total*" differentials of such functions considered as functions of an undetermined variable.

PARTIAL DIFFERENTIATION OF EXPLICIT FUNCTIONS OF SEVERAL VARIABLES.

67. If u be a function of several variables x, y, z, &c., and if for one of them as x the quantity $x+h$ be substituted, the other variables being left unchanged, the resulting "difference" of u is called the "partial difference with regard to x of u", and is represented by $\Delta_{(x)}u$, the symbol $\Delta_{(x)}$ standing for the expression "partial difference with regard to x of". Similarly, $\Delta_{(y)}u$, $\Delta_{(z)}u$, &c. respectively represent the partial differences with regard to y, z, &c. of u.

Also, if $x+h$ be substituted for x in u, and the process pointed out in Art. 6. be completed, the other variables being left unchanged, the resulting quantity is called (as in Art. 66.) the "*partial differential with regard to x of u*", and is represented by $\partial_{(x)}u$, the symbol $\partial_{(x)}$ standing for the expression "*partial differential with regard to x of*". Similarly $\partial_{(y)}u$, $\partial_{(z)}u$, &c. respectively represent the partial differentials with regard to y, z, &c. of u.

Again, since $\partial_{(x)}u$ may contain x either with or without the other variables, it may itself have a partial differential with regard to x, which is called the "second partial differential with regard to x of u", and which, according to the above notation, will be represented by $\partial_{(x)}\partial_{(x)}u$, or more simply by $\partial^2_{(x)}u$.—Similarly, the n^{th} partial differential with regard to x of u, if there should be any such function, will be represented by $\partial^n_{(x)}u$, and the corresponding differentials with regard to y, z, &c. by $\partial^n_{(y)}u$, $\partial^n_{(z)}u$, &c.

But $\partial_{(x)}u$ may contain y, z, &c., and may therefore have partial differentials with regard to each of those quantities, which will be respectively represented by $\partial_{(y)}\partial_{(x)}u$, $\partial_{(z)}\partial_{(x)}u$, &c.; and these latter functions may also have partial differentials with regard to x, y, z, &c., which will be respectively represented by $\partial_{(x)}\partial_{(y)}\partial_{(x)}u$, $\partial^2_{(y)}\partial_{(x)}u$, $\partial_{(z)}\partial_{(y)}\partial_{(x)}u$, &c., and $\partial_{(x)}\partial_{(z)}\partial_{(x)}u$, $\partial_{(y)}\partial_{(z)}\partial_{(x)}u$, $\partial^2_{(z)}\partial_{(x)}u$, &c. And generally the m^{th} partial differential with regard to x of the n^{th} partial differential with regard to y of the p^{th} partial differential with regard to z of the function u will be represented by $\partial^m_{(x)}\partial^n_{(y)}\partial^p_{(z)}u$.

Thus, if $u = x^2y^3$,

$$\partial_{(x)}u = 2xy^3, \text{ (Arts. 6 and 14.)}$$

$$\partial_{(y)}u = 3x^2y$$

$$\partial^2_{(x)}u = \partial_{(x)}(2xy^3) = 2y^3$$

$$\partial_{(x)}\partial_{(y)}u = \partial_{(x)}(3x^2y) = 6xy$$

$$\partial_{(y)}\partial^2_{(x)}u = \partial_{(y)}(2y^3) = 6y^2; \text{ and so on.}$$

It will be shewn in a subsequent proposition that the order in which the above processes of differentiation are performed is immaterial.

68. **Axiom.** If x, y, z, &c. h, k, l, &c. be quantities which are independent of each other, or are treated as such, any equation, which subsists between them or any functions of them when h, k, l, &c. are indeterminate, will continue to subsist when 0 is substituted for all or any of the quantities h, k, l, &c., provided that by such substitution no term becomes indefinite.

69. *If $f(x, y)$ be a function of x and y, its partial differential with regard to x when $y+k$ is substituted for y will be the same whether the substitution is made before or after the differentiation.*

For, if the substitution be made before the differentiation, the function becomes $f(x, \overline{y+k})$, and since the partial differentials of the functions $f(x, \overline{y+k})$, and $f(x, y)$ with regard to x are found by treating $y+k$ and y as constants, it is evident that the partial differential of $f(x, \overline{y+k})$, or $\partial_{(x)} f(x, \overline{y+k})$, is the same function of x and $y+k$, that the partial differential of $f(x, y)$, or $\partial_{(x)} f(x, y)$, is of x and y; and therefore that this latter quantity becomes identical with the former, when $y+k$ is substituted in it for y.

Thus, $\partial_{(x)}(x^2.a^y) = 2x.a^y$, which becomes $2x.a^{y+k}$ when $y+k$ is substituted for y. Also $\partial_{(x)}(x^2.a^{y+k}) = 2x.a^{y+k}$; i. e. the result is the same whether the substitution be made before or after the differentiation.

70. *If u be a function of x and y, and h and k be arbitrary and independent increments of x and y respectively, $\frac{\Delta_{(x)}\Delta_{(y)}u}{hk}$ becomes equal to $\partial_{(x)}\partial_{(y)}u$ when 0 is substituted for both h and k.*

Let $u = f(x, y)$, then

$\Delta_{(y)}u$ is $f(x, \overline{y+k}) - f(x, y)$

$\Delta_{(x)}\Delta_{(y)}u$ is $f(\overline{x+h}, \overline{y+k}) - f(\overline{x+h}, y) - \{f(x, \overline{y+k}) - f(x, y)\}$

therefore,

$$\frac{\Delta_{(x)}\Delta_{(y)}u}{hk} = \frac{1}{h}\left\{\frac{f(\overline{x+h}, \overline{y+k}) - f(\overline{x+h}, y)}{k} - \frac{f(x, \overline{y+k}) - f(x, y)}{k}\right\} (1).$$

If now 0 be substituted for k, no change being made in h, the part within the brackets $\{\ \}$ becomes

$$= \partial_{(y)} f(\overline{x+h}, y) - \partial_{(y)} f(x, y):$$

But $\partial_{(y)}f(x+h,\ y)$ is the partial differential with regard to y of $f(x,\ y)$ when $x+h$ is substituted for x before differentiation, and is therefore equal to the partial differential with regard to y of $f(x,\ y)$ in which such substitution has been made after differentiation (Art. 69). Hence, the above quantity is the partial "difference" with regard to x of the function $\partial_{(y)}f(x,\ y)$;

$$\text{and } \therefore\ \frac{1}{h}\{\partial_{(y)}f(\overline{x+h},\ y)-\partial_{(y)}f(x,\ y)\} \text{ becomes } =\partial_{(x)}\partial_{(y)}u,$$

when 0 is substituted for h.

But since the result arising from the substitution of 0 for both h and k in any function must be the same whether those substitutions are made together or successively, it follows that upon such substitutions being made the right side of equation (1) becomes $=\partial_{(x)}\partial_{(y)}f(x,\ y)$ or $\partial_{(x)}\partial_{(y)}u$;

$$\therefore\ \frac{\Delta_{(x)}\Delta_{(y)}u}{hk} \text{ becomes } = \partial_{(x)}\partial_{(y)}u,$$

when 0 is substituted for both h and k.

71. *In finding partial differentials of a function of several variables, the order in which the processes of differentiation are performed is immaterial.*

Let $u=f(x,\ y)$,

then $\Delta_{(y)}u$ is $f(x,\ \overline{y+k})-f(x,\ y)$

$$\begin{aligned}\Delta_{(x)}\Delta_{(y)}u \text{ is } & f(\overline{x+h},\ \overline{y+k})-f(\overline{x+h},\ y)-\{f(x,\ \overline{y+k})-f(x,\ y)\}\\ &=f(\overline{x+h},\ \overline{y+k})-f(x,\ \overline{y+k})-\{f(\overline{x+h},\ y)-f(x,\ y)\}\\ &=\Delta_{(y)}\Delta_{(x)}u;\end{aligned}$$

$$\therefore\ \frac{\Delta_{(x)}\Delta_{(y)}u}{hk}=\frac{\Delta_{(y)}\Delta_{(x)}u}{hk},$$

an equation subsisting between x, y, h and k, when h and k are indeterminate; and since the quantities on each side of the equation become respectively $\partial_{(x)}\partial_{(y)}u$ and $\partial_{(y)}\partial_{(x)}u$, when 0 is substituted for both h and k, each side of the equation must be capable of being reduced to such a form that when 0 is substituted for h and k, no term becomes indefinite, and therefore (Art. 68.) the equation will continue to subsist when such substitution is made;

$$\therefore\ \partial_{(x)}\partial_{(y)}u = \partial_{(y)}\partial_{(x)}u.$$

Hence also,
$$\begin{aligned}\partial^2_{(x)}\partial_{(y)}u &= \partial_{(x)}\partial_{(y)}\partial_{(x)}u\\ &= \partial_{(y)}\partial_{(x)}\partial_{(x)}u\\ &= \partial_{(y)}\partial^2_{(x)}u.\end{aligned}$$

And the same reasoning may evidently be extended to higher orders of differentials, and to functions of a greater number of variables.

Ex. 1. Let $u = x^m.y^n$.

$$\begin{aligned}\partial_{(x)}u &= mx^{m-1}.y^n;\\ \therefore\ \partial_{(y)}\partial_{(x)}u &= \partial_{(y)}(mx^{m-1}.y^n)\\ &= mnx^{m-1}.y^{n-1}.\end{aligned}$$

Again,
$$\begin{aligned}\partial_{(y)}u &= nx^m.y^{n-1};\\ \therefore\ \partial_{(x)}\partial_{(y)}u &= \partial_{(x)}(nx^m.y^{n-1})\\ &= mnx^{m-1}.y^{n-1} = \partial_{(y)}\partial_{(x)}u\end{aligned}$$

Ex. 2. Let $u = x.\sin y + y.\sin x$.

$$\begin{aligned}\partial_{(x)}u &= \sin y + y.\cos x\\ \partial_{(y)}u &= \sin x + x.\cos y\\ \partial_{(y)}\partial_{(x)}u &= \cos y + \cos x = \partial_{(x)}\partial_{(y)}u.\end{aligned}$$

Ex. 3. Let $u = xyz$.

$$\begin{aligned}\partial_{(x)}u &= yz\\ \partial_{(y)}u &= xz\\ \partial_{(z)}u &= xy\end{aligned}$$

$$\partial_{(y)}\partial_{(x)}u = z = \partial_{(x)}\partial_{(y)}u;$$
$$\partial_{(z)}\partial_{(y)}u = x = \partial_{(y)}\partial_{(z)}u;$$
$$\partial_{(z)}\partial_{(x)}u = y = \partial_{(x)}\partial_{(z)}u;$$
$$\therefore\ \partial_{(z)}\partial_{(y)}\partial_{(x)}u = 1 = \partial_{(z)}\partial_{(x)}\partial_{(y)}u = \partial_{(x)}\partial_{(z)}\partial_{(y)}u = \partial_{(x)}\partial_{(y)}\partial_{(z)}u$$
$$= \partial_{(y)}\partial_{(z)}\partial_{(x)}u = \partial_{(y)}\partial_{(x)}\partial_{(z)}u.$$

TOTAL DIFFERENTIATION OF EXPLICIT FUNCTIONS OF SEVERAL VARIABLES.

72. *To find the "total differential" of an explicit function of several variables, considered as arbitrary functions of an undetermined variable, in terms of the several "partial differentials" of the function with regard to the given variables and the differentials of the variables themselves with regard to the undetermined variable.*

Let u be a function of x and y, then, if x and y be considered as functions of an undetermined variable, u may be treated as a function of the same variable.

Now, ∂u the total differential of u with regard to the undetermined variable found by the methods given in the preceding sections will obviously consist of quantities multiplied by either ∂x or ∂y, the differentials of x and y; and if such quantities respectively be collected together an equation will be obtained of the form

$$\partial u = P.\partial x + Q.\partial y,$$

P and Q being functions of x and y only.

Again, the partial differential of u considered as a function of x only is found by treating y as a constant, wherever it occurs in the expression u. Hence, this partial differential must be equal to the right side of the above equation when 0 is substituted for ∂y, i. e. to $P.\partial x$. But (Art. 20.) the same partial differential of u is equal to the product of the differential of u considered as a function of x only and taken with regard to x, (or the partial differential of u with regard to x), and the differential of x with regard to the undetermined variable, i. e. to $\partial_{(x)}u.\partial x$,

$$\therefore\ \partial_{(x)}u.\partial x = P.\partial x,$$

$$\text{and } \partial_{(x)}u = P.$$

$$\text{Similarly, } \partial_{(y)}u = Q;$$

$$\therefore\ \partial u = \partial_{(x)}u.\partial x + \partial_{(y)}u.\partial y.$$

In the same manner, if u be a function of a greater number of variables, x, y, z, &c. it may be shewn that

$$\partial u = \partial_{(x)}u.\partial x + \partial_{(y)}u.\partial y + \partial_{(z)}u.\partial z + \text{\&c.}$$

in which equation, if x, y, z, &c. are independent of each other, the differentials ∂x, ∂y, ∂z, &c. are both arbitrary and independent; but if x, y, z, &c. are dependent on each other, the above differentials also, though arbitrary, will depend on each other.

Cor. It is sometimes convenient to take the differentials with regard to one of the quantities, as x, occurring in the function; in that case, since $\partial_x x = 1$, the above equation, by Art. 21, becomes

$$\partial_x u = \partial_{(x)}u + \partial_{(y)}u.\partial_x y + \partial_{(z)}u.\partial_x z + \text{\&c.}$$

in which, if x, y, z, &c. are independent of each other, $\partial_x y$, $\partial_x z$, &c. are arbitrary and independent.

Ex. 1. Let $u = x^y$; required $\partial_x u$.

$$\partial_{(x)}u = y.x^{y-1}, \quad \text{(Art 27.)}$$

$$\partial_{(y)}u = \log_\epsilon x.x^y, \quad \text{(Art. 31.)}$$

$$\begin{aligned}\therefore\ \partial_x u &= \partial_{(x)}u + \partial_{(y)}u.\partial_x y \\ &= y.x^{y-1} + \log_\epsilon x.x^y.\partial_x y \\ &= x^{y-1}.\{y + \log_\epsilon x.x.\partial_x y\}.\end{aligned}$$

Ex. 2. Let $u = xyz$; required ∂u.

$$\partial_{(x)} u = yz$$

$$\partial_{(y)} u = xz$$

$$\partial_{(z)} u = xy;$$

$$\therefore\ \partial u = \partial_{(x)} u . \partial x + \partial_{(y)} u . \partial y + \partial_{(z)} u . \partial z;$$

$$\therefore\ = yz . \partial x + xz . \partial y + xy . \partial z.$$

Ex. 3. $z = x^3 + y^3 - 3axy$; required $\partial_x z$.

$$\partial_{(x)} z = 3x^2 - 3ay$$

$$\partial_{(y)} z = 3y^2 - 3ax;$$

$$\therefore\ \partial_x z = \partial_{(x)} z + \partial_{(y)} z . \partial_x y$$

$$= 3(x^2 - ay) + 3(y^2 - ax) . \partial_x y.$$

Ex. 4. $z = c . \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$; required $\partial_x z$.

$$\partial_{(x)} z = -\frac{c}{a^2} . \frac{x}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} = -\frac{c^2 x}{a^2 z},$$

$$\partial_{(y)} z = -\frac{c}{b^2} . \frac{y}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} = -\frac{c^2 y}{b^2 z};$$

$$\therefore\ \partial_x z = -\frac{c^2 x}{a^2 z} - \frac{c^2 y}{b^2 z} . \partial_x y.$$

Ex. 5. $u = ad \sin \phi + bc . \sin \theta$, in which θ and ϕ are dependent variables connected by the equation $ad . \cos \phi - bc . \cos \theta = m$; required $\partial_\theta u$.

$$\partial_{(\theta)} u = bc . \cos \theta,$$

$$\partial_{(\phi)} u = ad . \cos \phi;$$

$$\therefore\ \partial_\theta u = bc . \cos \theta + ad . \cos \phi . \partial_\theta \phi.$$

But $bc.\sin\theta - ad.\sin\phi.\partial_\theta\phi = 0$;

$$\therefore\ \partial_\theta\phi = \frac{bc}{ad}.\frac{\sin\theta}{\sin\phi};$$

$$\text{and}\ \therefore\ \partial_\theta u = bc.\cos\theta + bc.\cos\phi.\frac{\sin\theta}{\sin\phi}$$

$$= bc.\frac{\sin(\theta+\phi)}{\sin\phi}.$$

73. *To find the total differentials of higher orders of an explicit function of several variables, considered as arbitrary functions of an undetermined variable, in terms of the partial differentials of the function with regard to the given variable and the differentials of the variables themselves with regard to the undetermined variable.*

Let u be a function of x and y, then

$$\partial u = \partial_{(x)}u.\partial x + \partial_{(y)}u.\partial y,\quad \text{(Art. 72.)}$$

$$\therefore\ \partial^2 u = \partial(\partial_{(x)}u.\partial x) + \partial(\partial_{(y)}u.\partial y),\quad \text{(Art. 22.)}$$

$$= \partial_{(x)}u.\partial\partial x + \partial x.\partial\partial_{(x)}u$$

$$+ \partial_{(y)}u.\partial\partial y + \partial y.\partial\partial_{(y)}u,\quad \text{(Art. 23.)}$$

Now x and y being considered as functions of the undetermined variable, $\partial\partial x$ is $\partial^2 x$, and $\partial\partial y$ is $\partial^2 y$. Also $\partial_{(x)}u$ and $\partial_{(y)}u$ being functions of x and y,

$$\partial\partial_{(x)}u = \partial^2_{(y)}u.\partial x + \partial_{(y)}\partial_{(x)}u.\partial y$$

$$\partial\partial_{(y)}u = \partial^2_{(y)}u.\partial y + \partial_{(x)}\partial_{(y)}u.\partial x.$$

Hence, substituting these values in the above equation,

$$\partial^2 u = \partial_{(x)}u.\partial^2 x + \partial^2_{(x)}u.(\partial x)^2 + \partial_{(y)}\partial_{(x)}u.\partial x.\partial y$$

$$+ \partial_{(y)}u.\partial^2 y + \partial^2_{(y)}u.(\partial y)^2 + \partial_{(x)}\partial_{(y)}u.\partial x.\partial y.$$

And since $\partial_{(y)}\partial_{(x)}u = \partial_{(x)}\partial_{(y)}u$, (Art. 71.)

$$\partial^2 u = \partial^2_{(x)}u.(\partial x)^2 + 2.\partial_{(x)}\partial_{(y)}u.\partial x.\partial y + \partial^2_{(y)}u.(\partial y)^2$$

$$+ \partial_{(x)}u.\partial^2 x + \partial_{(y)}u.\partial^2 y.$$

Cor. 1. If the differentials be taken with regard to one of the variables in the function, as x, since $\partial_x x = 1$, and $\partial_x^2 x = 0$, the above equation becomes, (Art. 21.)

$$\partial_x^2 u = \partial_{(x)}^2 u + 2.\partial_{(x)}\partial_{(y)} u.\partial_x y + \partial_{(y)}^2 u.(\partial_x y)^2 + \partial_{(y)} u.\partial_x^2 y.$$

Cor. 2. If x and y be considered as arbitrary *simple* functions of the undetermined variable, or of one of the variables, as x, occurring in the proposed function, ∂x, ∂y and $\partial_x y$ are constants, and therefore $\partial^2 x$, $\partial^2 y$, and $\partial_x^2 y$ are respectively equal to 0.

Hence on this supposition the above equations become

$$\partial^2 u = \partial_{(x)}^2 u.(\partial x)^2 + 2.\partial_{(x)}\partial_{(y)} u.\partial x.\partial y + \partial_{(y)}^2 u.(\partial y)^2,$$

$$\partial_x^2 u = \partial_{(x)}^2 u + 2.\partial_{(x)}\partial_{(y)} u.\partial_x y + \partial_{(y)}^2 u.(\partial_x y)^2.$$

But this supposition cannot be made unless the variables x and y are either independent, or simple functions, of each other. They may, however, be connected by any homogeneous equation; since the roots of such an equation solved as to one variable are always *simple* functions of the other.

The above methods may evidently be extended to functions of a greater number of variables, and to higher orders of differentials.

Thus, if $u = f(x, y, z)$;

$$\partial u = \partial_{(x)} u.\partial x + \partial_{(y)} u.\partial y + \partial_{(z)} u.\partial z, \quad \text{(Art. 72)}:$$

$$\begin{aligned} \therefore\ \partial^2 u &= \partial(\partial_{(x)} u.\partial x) + \partial(\partial_{(y)} u.\partial y) + \partial(\partial_{(z)} u.\partial z) \\ &= \partial_{(x)} u.\partial^2 x + \partial x.\partial\partial_{(x)} u \\ &\quad + \partial_{(y)} u.\partial^2 y + \partial y.\partial\partial_{(y)} u \\ &\quad + \partial_{(z)} u.\partial^2 z + \partial z.\partial\partial_{(z)} u. \end{aligned}$$

And,

$$\begin{aligned} \partial\partial_{(x)} u &= \partial_{(x)}^2 u.\partial x + \partial_{(y)}\partial_{(x)} u.\partial y + \partial_{(z)}\partial_{(x)} u.\partial z \\ \partial\partial_{(y)} u &= \partial_{(y)}^2 u.\partial y + \partial_{(x)}\partial_{(y)} u.\partial x + \partial_{(z)}\partial_{(y)} u.\partial z \\ \partial\partial_{(z)} u &= \partial_{(z)}^2 u.\partial z + \partial_{(x)}\partial_{(z)} u.\partial x + \partial_{(y)}\partial_{(z)} u.\partial y; \end{aligned}$$

$$\begin{aligned}\therefore\ \partial^2 u = {} & \partial_{(x)}u.\partial^2 x + \partial_{(y)}u.\partial^2 y + \partial_{(z)}u.\partial^2 z \\ & + 2.\partial_{(x)}\partial_{(y)}u.\partial x.\partial y + 2.\partial_{(x)}\partial_{(z)}u.\partial x.\partial z + 2.\partial_{(y)}\partial_{(z)}u.\partial y.\partial z \\ & + \partial^2_{(x)}u.(\partial x)^2 + \partial^2_{(y)}u.(\partial y)^2 + \partial^2_{(z)}u.(\partial z)^2.\end{aligned}$$

And the methods employed in the preceding corollaries, may be similarly introduced here also.

74. From Arts. 72, and 73, may be deduced a method of differentiating implicit functions of one variable differing from that employed in Art. 43, and possessing some advantages in particular cases.

Thus, if y be an implicit function of x, given by the equation $f(x, y) = 0$, and $f(x, y)$ be represented by u; by (Art. 72.)

$$\partial_x u = \partial_{(x)}u + \partial_{(y)}u.\partial_x y.$$

But (Art. 42.) $\partial_x u = 0$;

$$\therefore\ 0 = \partial_{(x)}u + \partial_{(y)}u.\partial_x y,$$

$$\text{and } \partial_x y = -\frac{\partial_{(x)}u}{\partial_{(y)}u}.$$

Again, by Art. 73,

$$\partial^2_x u = \partial^2_{(x)}u + 2.\partial_{(x)}\partial_{(y)}u.\partial_x y + \partial^2_{(y)}u.(\partial_x y)^2 + \partial_{(y)}u.\partial^2_x y.$$

But (Art. 42.) $\partial^2_x u = 0$;

$$\therefore\ \partial^2_x y = -\frac{1}{\partial_{(y)}u}\{\partial^2_{(x)}u + 2.\partial_{(x)}\partial_{(y)}u.\partial_x y + \partial^2_{(y)}u.(\partial_x y)^2\};$$

or, substituting the value of $\partial_x y$ found above,

$$\partial^2_x y = -\frac{1}{(\partial_{(y)}u)^3}\{\partial^2_{(x)}u.(\partial_{(y)}u)^2 - 2.\partial_{(x)}\partial_{(y)}u.\partial_{(x)}u.\partial_{(y)}u + \partial^2_{(y)}u.(\partial_{(x)}u)^2\};$$

and so on for the differentials of higher orders.

Ex. $x^3 + y^3 - 3axy = 0$. Required $\partial_x y$ and $\partial_x^2 y$.

$$\partial_{(x)} u = 3x^2 - 3ay$$

$$\partial_{(y)} u = 3y^2 - 3ax;$$

$$\therefore\ \partial_x y = -\frac{\partial_{(x)} u}{\partial_{(y)} u} = -\frac{x^2 - ay}{y^2 - ax}.$$

Again, $\partial_{(x)}^2 u = 6x$

$$\partial_{(y)}^2 u = 6y$$

$$\partial_{(x)} \partial_{(y)} u = -3a;$$

$$\therefore\ \partial_x^2 y = -\frac{1}{27(y^2 - ax)^3}.\{6x.9(y^2 - ax)^2 + 6a.9(x^2 - ay)(y^2 - ax) + 6y.9(x^2 - ay)^2\}$$

$$= -\frac{2}{(y^2 - ax)^3}.\{xy(x^3 + y^3 - 3axy + a^3)\}$$

$$= -\frac{2a^3xy}{(y^2 - ax)^3}.$$

DIFFERENTIATION OF IMPLICIT FUNCTIONS OF SEVERAL VARIABLES.

75. *To find the partial differentials of an implicit function of several variables.*

Let z be an implicit function of x and y given by the equation $u = 0$, u being a function of x, y and z. Then, if, in the function u, y be treated as a constant, and z as a function of x only, u becomes a function of two variables x and z only, and its differential with regard to x on this supposition will be equal to 0. (Art. 42.)

But this differential $= \partial_{(x)} u + \partial_{(z)} u . \partial_{(x)} z$, (Art. 72.) $\partial_{(x)} z$ being substituted for $\partial_x z$, since z, though treated as a function of x only, is in fact a function of x and y.

$$\text{Hence}\quad \partial_{(x)}u + \partial_{(z)}u.\partial_{(x)}z = 0;$$

$$\therefore\ \partial_{(x)}z = -\frac{\partial_{(x)}u}{\partial_{(z)}u}.$$

$$\text{Similarly,}\ \partial_{(y)}z = -\frac{\partial_{(y)}u}{\partial_{(z)}u}.$$

And it is evident that, whatever be the number of the quantities of which z is a function, the partial differentials of z with regard to such quantities may be determined by equations of a similar form.

Ex. 1. $xyz = a^3$. Required $\partial_{(x)}z$, and $\partial_{(y)}z$.

$$\partial_{(x)}u \text{ or } \partial_{(x}\,(xyz - a^3) = yz$$

$$\partial_{(y)}u = xz$$

$$\partial_{(z)}u = xy;$$

$$\therefore\ \partial_{(x)}z = -\frac{yz}{xy}$$

$$= -\frac{z}{x}.$$

$$\text{And}\ \partial_{(y)}z = -\frac{xz}{xy}$$

$$= -\frac{z}{y}.$$

Ex. 2. $(x-az)^2 + (y-bz)^2 = 2r.(x-az)$, [the equation to the surface of an oblique cylinder] required $\partial_{(x)}z$, and $\partial_{(y)}z$.

$$u = (x-az)^2 + (y-bz)^2 - 2r.(x-az) = 0;$$

$$\therefore\ \partial_{(x)}u = 2\,(x-az) - 2r$$

$$\partial_{(y)}u = 2\,(y-bz)$$

$$\partial_{(z)}u = -2a\,(x-az) - 2b\,(y-bz) + 2ar;$$

$$\therefore\ \partial_{(x)}z = \frac{x-az-r}{a\,(x-az)+b\,(y-bz)-ar}.$$

$$\text{And}\ \partial_{(y)}z = \frac{y-bz}{a\,(x-az)+b\,(y-bz)-ar}.$$

Ex. 3. $a^x.b^y.c^z = A$, required $\partial_{(x)}z$, and $\partial_{(y)}z$.

$$u = a^x.b^y.c^z - A = 0;$$

$$\therefore\ \partial_{(x)}u = a^x.b^y.c^z.\log_\epsilon a$$

$$\partial_{(y)}u = a^x.b^y.c^z.\log_\epsilon b$$

$$\partial_{(z)}u = a^x.b^y.c^z.\log_\epsilon c;$$

$$\therefore\ \partial_{(x)}z = -\frac{\log_\epsilon a}{\log_\epsilon c} = -\frac{\log a}{\log c}.$$

$$\text{And } \partial_{(y)}z = -\frac{\log_\epsilon b}{\log_\epsilon c} = -\frac{\log b}{\log c}.$$

76. *To find the total differential of an implicit function of several variables with regard to any one of the variables.*

Let z be an implicit function of x and y given by the equation $u = 0$, u being a function of x, y, and z.

Then, by the preceding proposition,

$$\partial_{(x)}u + \partial_{(z)}u.\partial_{(x)}z = 0 \ldots\ldots\ldots (1)$$

$$\text{and } \partial_{(y)}u + \partial_{(z)}u.\partial_{(y)}z = 0.$$

Multiplying the latter equation by $\partial_x y$, the total differential with regard to x of y,

$$\partial_{(y)}u.\partial_x y + \partial_{(z)}u.\partial_{(y)}z.\partial_x y = 0 \ldots\ldots\ldots (2).$$

And adding together (1) and (2),

$$\partial_{(x)}u + \partial_{(y)}u.\partial_x y + \partial_{(z)}u.\{\partial_{(x)}z + \partial_{(y)}z.\partial_x y\} = 0.$$

But $\partial_{(x)}z + \partial_{(y)}z.\partial_x y = \partial_x z$, the total differential with regard to x of z;

$$\therefore\ \partial_{(x)}u + \partial_{(y)}u.\partial_x y + \partial_{(z)}u.\partial_x z = 0;$$

from which equation the total differential of z with regard to x may be found in terms of x, y, z, and constants, and the total differential with regard to x of y.

Cor. If x, y, and z be treated as functions of an undetermined variable,

$$\partial_x y = \frac{\partial y}{\partial x}, \text{ and } \partial_x z = \frac{\partial z}{\partial x};$$

$$\text{therefore } \partial_{(x)} u . \partial x + \partial_{(y)} u . \partial y + \partial_{(z)} u . \partial z = 0;$$

which determines the relation subsisting between the variables and their total differentials with regard to the undetermined variable.

It is evident that the method of finding the above equations may be extended to functions of any greater number of variables.

Obs. If the only relations which subsist between x, y, and z are given by, or can be determined from, the equation $u = 0$, it is evident that in finding $\partial_x z$ from the former of the above equations, the quantity $\partial_x y$ is perfectly arbitrary; and that, in finding ∂z from the latter equation, ∂x and ∂y are perfectly arbitrary and also independent.

Ex. 1. $xy + xz + yz = c$; required the equation between x, y, z and their total differentials.

$$u = xy + xz + yz - c = 0;$$

$$\therefore \partial_{(x)} u = y + z$$

$$\partial_{(y)} u = x + z$$

$$\partial_{(z)} u = x + y,$$

$$\text{and } \partial_{(x)} u . \partial x + \partial_{(y)} u . \partial y + \partial_{(z)} u . \partial z = 0;$$

$$\therefore (y + z) . \partial x + (x + z) . \partial y + (x + y) . \partial z = 0.$$

Ex. 2. $\dfrac{ay - bz}{cz - ax} = d.$

$$u = \frac{ay - bz}{cz - ax} - d = 0;$$

$$\therefore\ \partial_{(x)}u = \frac{a.(ay - bz)}{(cz - ax)^2}$$

$$\partial_{(y)}u = \frac{a}{cz - ax}$$

$$\partial_{(z)}u = \frac{a.(bx - cy)}{(cz - ax)^2};$$

$$\therefore\ (ay - bz).\partial x + (cz - ax).\partial y + (bx - cy).\partial z = 0.$$

Obs. In applying the above rules to particular examples the results will obviously be the same, whether the values of $\partial_{(x)}u$, $\partial_{(y)}u$, $\partial_{(z)}u$ be found separately and be substituted in the general equations, or whether the given equation be differentiated on the supposition that two of the variables are functions of the third or all of them functions of an undetermined variable, and the coefficients of the respective differentials be collected together.

If, as is frequently the case, in the quantity to be differentiated there should occur an arbitrary function, as ϕr, its partial differential with regard to any variable, as x, of which it is a function, will be found from the equation $\partial_{(x)}\phi r = \partial_{(r)}\phi r.\partial_{(x)}r$, (Art. 15).

Thus, if $z = \phi\sqrt{x^2+y^2}$, let $\sqrt{x^2+y^2}$ be represented by r,

$$\text{then } z = \phi r$$

$$\text{and } \partial_{(x)}z = \partial_{(r)}\phi r.\partial_{(x)}r$$

$$= \partial_{(r)}\phi r.\frac{x}{r}.$$

$$\text{Similarly } \partial_{(y)}z = \partial_{(r)}\phi r.\frac{y}{r}.$$

SECTION VII.

THE DIFFERENTIATION OF CERTAIN GEOMETRICAL AND PHYSICAL FUNCTIONS OF SEVERAL VARIABLES.

77. Axiom I. If $\phi(\overline{x+h}, \overline{y+k})$ increase or decrease continuously for the successive values of h and k between 0 and some assumed definite quantity, the difference between $\phi(\overline{x+h}, \overline{y+k})$ and $\phi(x, y)$ can, by diminishing h and k, be made to decrease continuously and become less than any assignable quantity.

78. Axiom II. If the difference between $\dfrac{\Delta_{(x)}\Delta_{(y)}u}{hk}$ and any given function of x and y, as $\phi(x, y)$, can, by diminishing h, be made to decrease continuously and become less than any assignable quantity, $\dfrac{\Delta_{(x)}\Delta_{(y)}u}{hk} = \phi(x, y)$, when 0 is substituted for both h and k.

79. *If* $\dfrac{\Delta_{(x)}\Delta_{(y)}u}{hk}$ *is equal to* $\phi(x, y)$, *when* 0 *is substituted for both* h *and* k, $\phi(x, y)$ *is, or is equal to, the differential with regard to* x *of the differential with regard to* y *of* u, *or* $\partial_{(x)}\partial_{(y)}u$.

For, when 0 is substituted for both h and k,

$$\frac{\Delta_{(x)}\Delta_{(y)}u}{hk} \text{ becomes equal to } \partial_{(x)}\partial_{(y)}u, \text{ (Art. 70).}$$

Hence $\phi(x, y) = \partial_{(x)}\partial_{(y)}u.$

80. *If* $\Delta_{(x)}\Delta_{(y)}u$ *be between* $hk.\phi(x,\, y)$ *and* $hk.\phi(\overline{x+h},\, \overline{y+k})$ *for all values of* h *and* k *between* 0 *and some assumed definite quantities, and* $\phi(\overline{x+h},\, \overline{y+k})$ *increase or decrease continuously between those limits of* h *and* k, $\partial_{(x)}\partial_{(y)}u = \phi(x,\, y)$.

For it follows from the hypothesis, that

$$\Delta_{(x)}\Delta_{(y)}u \sim hk.\phi(x,\, y)$$

$$= \text{a quantity not} > hk.\{\phi(\overline{x+h},\, \overline{y+k}) \sim \phi(x,\, y)\};$$

$$\text{and } \therefore\ \frac{\Delta_{(x)}\Delta_{(y)}u}{hk} \sim \phi(x,\, y)$$

$$= \text{a quantity not} > \{\phi(\overline{x+h},\, \overline{y+k}) \sim \phi(x,\, y)\},$$

for all values of h and k between 0 and the assumed quantities; and by hypothesis $\phi(\overline{x+h},\, \overline{y+k})$ increases or decreases continuously between those limits of h and k.

Hence, by diminishing h and k (Art. 77.)

$$\phi(\overline{x+h},\, \overline{y+k}) \sim \phi(x,\, y),$$

and therefore $\dfrac{\Delta_{(x)}\Delta_{(y)}u}{hk} \sim \phi(x,\, y)$ may be made to decrease continuously and become less than any assignable quantity;

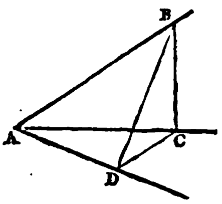
$$\therefore\ (\text{Art. 78.})\ \frac{\Delta_{(x)}\Delta_{(y)}u}{hk} = \phi(x,\, y),$$

when 0 is substituted for h and k,

$$\text{and } \therefore\ (\text{Art. 79.})\ \partial_{(x)}\partial_{(y)}u = \phi(x,\, y).$$

Note. The above Axioms and Propositions may be easily extended to functions of a greater number of variables.

81. LEMMA. *If a solid angle be contained by three plane angles, and two of the planes forming it be at right angles to each other, the cosine of the angle opposite to the right angle is equal to the product of the cosines of the other two.*



Let *A* be the solid angle contained by the plane angles *BAC*, *CAD*, *DAB*; and let the planes *BAC*, *CAD* be at right angles to each other. Take in *AB* any point *B*, and draw *BC* perpendicular to *AC*; then, since any plane passing through *BC* is perpendicular to the plane in which *AD* lies, there must be one such plane perpendicular to *AD*; let *BDC* be this plane; then *ADB*, *ADC* are right angles;

$$\therefore \cos BAD = \frac{AD}{AB}$$

$$= \frac{AD}{AC} \cdot \frac{AC}{AB}$$

$$= \cos CAD . \cos BAC.^*$$

82. *In a curved surface referred to three rectangular co-ordinates, to find the differential of the area of the* SURFACE *bounded by two of the co-ordinate planes and any two other planes parallel to the former.*

Let *AX*, *AY*, *AZ* be the axes of co-ordinates;

x, y, z the co-ordinates of any point *P* in the surface;

BKPM (*S*) the area of the surface bounded by the co-ordinate planes xz, yz, and by planes parallel to them through *P*; *KPR*, *MPQ* being the intersections of the latter planes with the surface; and let two other planes also be drawn parallel to these through a point *S* in the surface, so as not to cut the surface *BKPM*.

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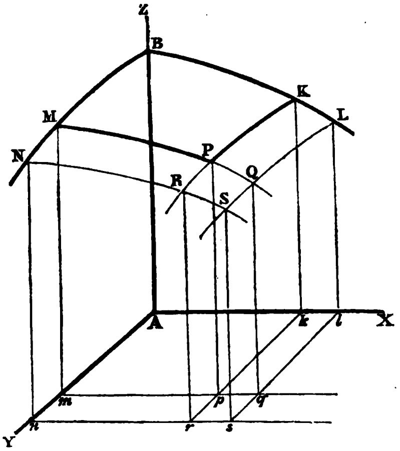
[*] One of Napier's Rules.

Then it is evident that the surface S is a function of the arcs MP, KP; and if these be represented by s and σ respectively, and PQ, PR by h and k respectively, it is also evident that

$$MPRN \text{ is } \Delta_{(\sigma)} S$$

and $PQSR$ is $\Delta_{(s)}\Delta_{(\sigma)} S$.



If now a line equal to the greatest section of the surface $PQSR$ by a plane parallel to xz be moved from PQ to RS in such a manner as always to have one of its extremities in the line PR, to lie in a plane parallel to xz, and to coincide at every point with the surface, it is evident that it will pass over a portion of the surface not less than $PQSR$.

Again, it is evident that the portion of the surface so passed over will not be greater than the parallelogram passed over by a straight line of equal length, moved parallel to itself along another straight line equal in length to the greatest section of $PQSR$ by a plane parallel to zy, and making with this latter line an angle equal to the greatest angle between any sections of $PQSR$ by planes parallel to zx, zy.

If therefore l and λ be the lengths of any sections of $PQSR$ by planes parallel to zx, zy, and θ the angle between them or any other similar sections, it is evident that $PQSR$ is not greater than the greatest value of $l.\lambda.\sin\theta$; and similarly it may be shewn that it is not less than the least value.

Now let β be the angle between PQ, PR at the point P, then

$$PQ.PR.\sin\beta \text{ is one of the values of } l.\lambda.\sin\theta;$$

$\therefore$ $PQSR \sim PQ.PR.\sin\beta$ is not greater than the difference between the greatest and least values of $l.\lambda.\sin\theta$;

$\therefore \dfrac{PQSR}{PQ.PR} \sim \sin\beta$ is not greater than the difference between the greatest and least values of $\dfrac{l}{PQ}.\dfrac{\lambda}{PR}.\sin\theta$.

But, by continuously diminishing PQ and PR, the difference between the greatest and least values of $\sin\theta$ continuously diminishes; and therefore, if $PQSR$ be taken such that the difference between the greatest and least values of $\dfrac{l}{PQ}$ and $\dfrac{\lambda}{PR}$ respectively also continuously diminishes,

$$\frac{PQSR}{PQ.PR} \sim \sin\beta, \text{ or } \frac{\Delta_{(s)}\Delta_{(\sigma)}S}{h.k} \sim \sin\beta,$$

can, by diminishing h and k, be made to decrease continuously and become less than any assignable quantity;

$$\therefore \partial_{(s)}\partial_{(\sigma)}S = \sin\beta.$$

Cor. If T, t be the angles which the tangents to the arcs PQ, PR at the point P make with xy, it will be easily seen that

$$\begin{aligned} \partial_{(x)}\partial_{(y)}S &= \partial_{(x)}s \,.\, \partial_{(y)}\sigma \,.\, \partial_{(s)}\partial_{(\sigma)}S \\ &= \sec T.\sec t.\sin\beta. \end{aligned}$$

Also by Lemma (Art. 81.)

$$\begin{aligned} \cos\beta &= \cos(90^0 - T).\cos(90^0 - t) \\ &= \sin T.\sin t; \end{aligned}$$

$$\begin{aligned} \therefore \sec T.\sec t.\sin\beta &= \sec T.\sec t.\sqrt{1 - (\sin T.\sin t)^2} \\ &= \sqrt{(\sec T)^2.(\sec t)^2 - (\tan T)^2.(\tan t)^2} \\ &= \sqrt{1 + (\tan T)^2 + (\tan t)^2}; \end{aligned}$$

$$\therefore \partial_{(x)}\partial_{(y)}S = \sqrt{1 + (\partial_{(x)}z)^2 + (\partial_{(y)}z)^2}.$$

Obs. The same values of $\partial_{(x)}\partial_{(y)}S$ will evidently be obtained, if S, instead of being bounded by the co-ordinate planes zx, zy, be bounded by any other planes parallel to them.

83. *In a body bounded by a curved surface referred to three rectangular co-ordinates, to find the differential of the* volume *bounded by the co-ordinate planes and any two other planes parallel to two of the former.*

Construct the fig. as in the preceding Proposition, and let V represent the volume Bp. Then it is evident that the volume Ps is $\Delta_{(x)}\Delta_{(y)}V$; and it is also evident that this volume is not greater than the parallelopiped whose base is $pqsr$ and altitude the greatest of the ordinates parallel to AZ for the several points in $PQSR$, and not less than the parallelopiped whose base is $pqsr$ and altitude the least of such ordinates.

But, if $\overline{kl} = h$, and $\overline{mn} = k$, $pqsr = h.k$;

$\therefore$ $\Delta_{(x)}\Delta_{(y)}V$ is not $> h.k. \times$ greatest ordinate for $PQSR$ $\parallel$ to AZ.

and not $< h.k \times$ least ordinate ....

$\therefore$ $\dfrac{\Delta_{(x)}\Delta_{(y)}V}{h.k}$ is not $>$ greatest ordinate..............................

and not $<$ least ....

But Pp, or z, is one of these ordinates;

$\therefore$ $\dfrac{\Delta_{(x)}\Delta_{(y)}V}{hk} \sim z$ is between the greatest and least of the above ordinates.

And if the points Q and R be taken such that, for every point, as S, in $PQSR$, the ordinate parallel to AZ increases or decreases continuously, the above difference may, by diminishing both h and k, be made to decrease continuously, and become less than any assignable quantity;

$\therefore$ $\Delta_{(x)}\Delta_{(y)}V = z$, when 0 is substituted for h and k;

$$\therefore \partial_{(x)}\partial_{(y)}V = z.$$

Cor. Since $\partial_{(x)}\partial_{(y)}V = z$; $\therefore$ $\partial_{(x)}\partial_{(y)}\partial_{(z)}V = 1$.

PHYSICAL FUNCTIONS.

84. *In a body of uniform density bounded by a curved surface referred to three rectangular co-ordinates, to find the differential of the* "moment,"* *about one of the axes of co-ordinates, of a portion of the body cut off by the co-ordinate planes and any two other planes parallel to the two co-ordinate planes which intersect in that axis.*

Construct the fig. as in Art. 82; and let M be the "moment" of the mass Bp about the axis AZ; then, $\Delta_{(x)}\Delta_{(y)}M$ is the "moment" of the mass Ps about the same axis. And it is evident, that this latter "moment" is greater than it would be, if the whole mass were collected at p, and less than if it were collected at s:—also that the mass Ps is greater than the parallelopiped whose base is $pqsr$ and altitude the least of the ordinates parallel to AZ for the several points in $PQSR$, and less than the parallelopiped whose base is $pqsr$ and altitude the greatest of such ordinates. Therefore, if ρ be the density of the body,

$$\Delta_{(x)}\Delta_{(y)}M \text{ is } > \rho.x.h.k \times \text{least ordinate for } PQSR \parallel \text{ to } AZ.$$

$$\text{and } < \rho.\overline{x+h}.h.k \times \text{greatest} \text{ ....}$$

But z is one of these ordinates;

$$\therefore\ \Delta_{(x)}\Delta_{(y)}M \sim \rho.x.h.k.z \text{ is } < \rho.\overline{x+h}.h.k \times \text{greatest ordinate}$$
$$\sim \rho.x.h.k \times \text{least ....}$$

$$\therefore\ \frac{\Delta_{(x)}\Delta_{(y)}M}{h.k} \sim \rho.x.z \quad \text{is } < \rho.\overline{x+h} \times \text{greatest ordinate}$$
$$\sim \rho.x \times \text{least ....}$$

Hence, by diminishing h and k,

$\dfrac{\Delta_{(x)}\Delta_{(y)}M}{h.k} \sim \rho.x.z$ may be made to decrease continuously and become less than any assignable quantity;

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* Vide page 73.

$\therefore\ \frac{\Delta_{(x)}\Delta_{(y)}M}{hk} = \rho.x.z$, when 0 is substituted for h and k;

$$\therefore\ \partial_{(x)}\partial_{(y)}M = \rho.x.z.$$

Cor. Hence also $\partial_{(x)}\partial_{(y)}\partial_{(z)}M = \rho.x$.

85. *In a body of uniform density bounded by a curved surface referred to three rectangular co-ordinates, to find the differential of the* "moment of inertia,"* *about one of the axes of co-ordinates, of a portion of the body cut off by the co-ordinate planes and any two other planes parallel to the two co-ordinate planes which intersect in that axis.*

Construct the fig. as in Art. 82, and let μ be the "moment of inertia" of the mass Bp about the axis AZ. Then proceeding in the same manner as in Art. 84, and supposing Ap, As joined, it may be shewn that

$\Delta_{(x)}\Delta_{(y)}\mu$ is $> \rho.\overline{Ap}^2.h.k \times$ least ordinate for $PQSR$ ∥ to AZ.

and $< \rho.\overline{As}^2.h.k \times$ greatest............................,

And that, by diminishing h and k,

$\frac{\Delta_{(x)}\Delta_{(y)}\mu}{h.k} \sim \rho.\overline{Ap}^2.z$ may be made to decrease continuously and become less than any assignable quantity.

Thus $\frac{\Delta_{(x)}\Delta_{(y)}\mu}{h.k}$, when 0 is substituted for h and k, becomes equal to $\rho.\overline{Ap}^2.z$;

$$\therefore\ \partial_{(x)}\partial_{(y)}\mu = \rho.(x^2 + y^2).z.$$

And $\partial_{(x)}\partial_{(y)}\partial_{(z)}\mu = \rho.(x^2 + y^2)$.

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* Vide page 75.

SECTION VIII.

CHANGE OF THE VARIABLE OF DIFFERENTIATION. ELIMINATION OF CONSTANTS AND FUNCTIONS. VALUES OF FUNCTIONS APPEARING UNDER THE FORM $\frac{0}{0}$.

I. CHANGE OF THE VARIABLE OF DIFFERENTIATION.[*]

EXPRESSIONS, which involve differentials taken with regard to a particular variable, may be transformed into equivalent ones, and complex expressions may sometimes be simplified, by *changing the variable of differentiation*, i. e. by substituting for such differentials other quantities equivalent to them involving differentials taken with regard to a different variable.

The following Propositions exhibit generally the method of determining the quantities to be so substituted.

86. *To find $\partial_x u$, $\partial_x^2 u$, $\partial_x^3 u$, &c. in terms of the differentials of u and x taken with regard to any other variable of which u and x are considered functions.*

$$(1) \qquad \partial_x u = \frac{\partial u}{\partial x} \quad \text{(Art. 20.)}$$

$$(2) \qquad \partial_x^2 u = \partial_x \frac{\partial u}{\partial x}$$

$$= \frac{1}{\partial x} . \partial \frac{\partial u}{\partial x}$$

$$= \frac{1}{\partial x} . \frac{\partial x . \partial^2 u - \partial u . \partial^2 x}{(\partial x)^2}$$

$$= \frac{\partial x . \partial^2 u - \partial u . \partial^2 x}{(\partial x)^3}$$

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[*] The "variable of differentiation" is sometimes called the "principal variable," and also sometimes the "independent variable."

$$(3) \qquad \partial_x^3 u = \partial_x \frac{\partial x.\partial^2 u - \partial u.\partial^2 x}{(\partial x)^3}$$

$$= \frac{1}{\partial x}.\partial \frac{\partial x.\partial^2 u - \partial u.\partial^2 x}{(\partial x)^3}$$

$$= \frac{1}{(\partial x)^5}\{\partial^3 u.(\partial x)^2 - 3.\partial^2 u.\partial^2 x.\partial x + 3.\partial u.(\partial^2 x)^2 - \partial u.\partial^3 x.\partial x\}.$$

And the same method may be extended to differentials of higher orders.

87. *To find $\partial_x u$, $\partial_x^2 u$, $\partial_x^3 u$, &c. in terms of the differentials of x taken with regard to u.*

$$(1) \qquad \partial_x u = \frac{1}{\partial_u x} \quad \text{(Art. 16.)}$$

$$(2) \qquad \partial_x^2 u = \partial_x \frac{1}{\partial_u x}$$

$$= \frac{1}{\partial_u x}.\partial_u \frac{1}{\partial_u x}$$

$$= -\frac{\partial_u^2 x}{(\partial_u x)^3}.$$

$$(3) \qquad \partial_x^3 u = \partial_x \left(-\frac{\partial_u^2 x}{(\partial_u x)^3}\right)$$

$$= \frac{1}{\partial_u x}.\partial_u \left(-\frac{\partial_u^2 x}{(\partial_u x)^3}\right)$$

$$= \frac{1}{(\partial_u x)^5}\{3.(\partial_u^2 x)^2 - \partial_u^3 x.\partial_u x\}.$$

And the same method may be extended to differentials of higher orders.

Obs. The values thus determined for $\partial_x u$, $\partial_x^2 u$, $\partial_x^3 u$, &c. may be deduced from those obtained in the preceding Article by substituting u for the undetermined variable, and observing that $\partial_u u = 1$, $\partial_u^2 u = 0$, $\partial_u^3 u = 0$, &c.

It is evident from the form of the equations in the two preceding Articles that the *order* of any expression, determined by the highest order of differentials contained in it, will not be altered by changing the variable of differentiation.

Ex. 1. Given $R = \dfrac{\{1+(\partial_x y)^2\}^{\frac{3}{2}}}{-\partial_x^2 y}$, required the equivalent general expression for R when the variable of differentiation is changed to *any* other.

$$\text{Since } \partial_x y = \frac{\partial y}{\partial x},$$

$$\text{and } \partial_x^2 y = \frac{1}{(\partial x)^3}.\{\partial x.\partial^2 y - \partial y.\partial^2 x\};$$

$$\therefore\ R = \frac{\{(\partial x)^2 + (\partial y)^2\}^{\frac{3}{2}}}{\partial y.\partial^2 x - \partial x.\partial^2 y}.$$

Ex. 2. Transform $\partial_x^2 y - x.\partial_x y + e^y.(\partial_x y)^3 = 0$ into the equivalent equation in which the differentials are taken with regard to y.

$$\text{Since } \partial_x^2 y = -\frac{\partial_y^2 x}{(\partial_y x)^3},$$

$$\text{and } \partial_x y = \frac{1}{\partial_y x};$$

$$\therefore\ -\frac{\partial_y^2 x}{(\partial_y x)^3} - x.\frac{1}{\partial_y x} + e^y.\frac{1}{(\partial_y x)^3} = 0,$$

$$\text{or } \partial_y^2 x - x.(\partial_y x)^2 + e^y = 0.$$

Ex. 3. Required the equation equivalent to

$$\partial_x^2 y - \frac{x}{1-x^2}.\partial_x y + \frac{y}{1-x^2} = 0$$

in which θ is the variable of differentiation and $x = \cos\theta$.

$$\partial_\theta x = -\sin\theta,$$

$$\partial_\theta^2 x = -\cos\theta;$$

$$\therefore\ \partial_x^2 y = \frac{1}{(\partial_\theta x)^3}\{\partial_\theta x.\partial_\theta^2 y - \partial_\theta y.\partial_\theta^2 x\}\quad \text{(Art. 86.)}$$

$$= \frac{1}{(\sin\theta)^3}\{\sin\theta.\partial_\theta^2 y - \cos\theta.\partial_\theta y\},$$

$$\text{and } \partial_x y = \frac{\partial_\theta y}{\partial_\theta x} = -\frac{\partial_\theta y}{\sin\theta};$$

$$\therefore\ \frac{1}{(\sin\theta)^3}\{\sin\theta.\partial_\theta^2 y - \cos\theta.\partial_\theta y\} + \frac{\cos\theta}{(\sin\theta)^3}.\partial_\theta y + \frac{y}{(\sin\theta)^2} = 0,$$

$$\text{or}\quad \partial_\theta^2 y + y = 0.$$

II. ELIMINATION OF CONSTANTS AND FUNCTIONS.

88. If $u = 0$ be an equation subsisting between the two variables x and y, then (Art. 42.) $u=0$, $\partial_x u=0$, $\partial_x^2 u=0$, &c. are equations which hold for the same values of x and y, and therefore any constant which appears in any two of these equations may be eliminated between them. In the same manner any n constants which appear in $\overline{n+1}$ of those equations may be eliminated between such equations.

Hence, generally, to eliminate n constants from any equation, u must be differentiated successively n times with regard to the same variable, and the n constants eliminated, in the usual manner, between the $\overline{n+1}$ equations $u = 0$, $\partial_x u = 0$, $\partial_x^2 u = 0 \ldots\ldots \partial_x^n u = 0$.

It may happen, from the particular form of the proposed equation, that the constants to be eliminated will disappear by differentiation *only;* and such a form may frequently be given to the equation without much trouble, by solving it with respect to the constant to be eliminated. Thus, if it be required to eliminate a from the equation $x^2 - 2ay - a^2 = 0$, we have $a^2 + 2ay + y^2 = x^2 + y^2$; $\therefore\ a + y = \sqrt{x^2 + y^2}$, and differentiating, $\partial_x y = \dfrac{x + y.\partial_x y}{\sqrt{x^2 + y^2}}$ an equation not involving a, and holding true for the same values of x and y as the proposed equation.

It may also happen that more than n constants can be eliminated by means of $n+1$ equations, but in such cases it will be found that two or more of the constants thus eliminated may be considered as one in the original equation.

Ex. Required to eliminate the "constants" a, b, r, from the equation $(x-a)^2 + (y-b)^2 = r^2$.

$$\text{Since } (x-a)^2 + (y-b)^2 - r^2 = 0 \text{............} (1)$$

$$x - a + (y-b).\partial_x y = 0 \text{............} (2)$$

$$\text{and } 1 + (\partial_x y)^2 + (y-b).\partial_x^2 y = 0 \text{............} (3);$$

from which latter equation a and r are eliminated.

Also, by (2) and (3)

$$x - a = \frac{\partial_x y}{\partial_x^2 y}.\{1 + (\partial_x y)^2\}; \text{............} (4)$$

from which b and r are eliminated.

And, again, by differentiating (3) and eliminating b between the resulting equation and (3), or by differentiating (4) only, all the *three* constants a, b, r, will disappear.

89. The elimination of particular functions occurring in the equation $u = f(x, y) = 0$ may also be effected by differentiation in nearly the same manner as the elimination of constants.

Thus, if $u = 0$ contain some *irrational* function of one of the variables (x) as $v^{\frac{m}{n}}$, $\partial_x u = 0$ will involve a new quantity $v^{\frac{m}{n}-1}$, or $v^{\frac{m}{n}}.\frac{1}{v}$. Hence, between the two equations $u = 0$, $\partial_x u = 0$, the quantity $v^{\frac{m}{n}}$ may be eliminated.

If the function to be eliminated from $u = 0$ be of the form a^v, or $\log v$, or any "inverse goniometrical function," as $\sin^{-1} v$, $\cos^{-1} v$, &c., no new function of the same class will be introduced by differentiation, and the original function will either not appear in $\partial_x u = 0$, or may be eliminated between $u = 0$ and $\partial_x u = 0$.

If $u = 0$ contain $\sin v$, $\partial_x u = 0$ will contain $\cos v$, for which $\sqrt{1-(\sin v)^2}$ may be substituted, and therefore $\sin v$ may be eliminated.—Similarly for any other goniometrical function.

By reasoning similar to the above it will appear that two of the above functions may be eliminated between the three equations $u=0$, $\partial_x u = 0$, $\partial_x^2 u = 0$; and, generally, n of such functions may be eliminated between the $\overline{n+1}$ equations $u=0$, $\partial_x u = 0$, $\partial_x^2 u = 0, \ldots \partial_x^n u = 0$.

Ex. 1. $u = ax - y\sqrt{a^2 + x^2} = 0$, (1), required to eliminate the irrational quantity.

$$\partial_x u = a - \partial_x y . \sqrt{a^2 + x^2} - \frac{xy}{\sqrt{a^2 + x^2}} = 0.$$

But by (1) $\sqrt{a^2 + x^2} = \dfrac{ax}{y}$;

$$\therefore\ a - \frac{ax}{y} . \partial_x y - \frac{y^2}{a} = 0.$$

Ex. 2. $u = y - \cos^{-1}\dfrac{y}{x} = 0$; to eliminate the inverse trigonometrical function.

$$\partial_x u = \partial_x y + \frac{\partial_x \dfrac{y}{x}}{\sqrt{1 - \left(\dfrac{y}{x}\right)^2}} = 0;$$

$$\therefore\ \partial_x y + \frac{y - x.\partial_x y}{x\sqrt{x^2 - y^2}} = 0.$$

Ex. 3. $y = \epsilon^x . \cos x$, to eliminate ϵ^x and $\cos x$.

$$\text{Since } y - \epsilon^x . \cos x = 0 \ldots\ldots\ldots\ldots\ldots (1);$$

$$\therefore\ \partial_x y - \epsilon^x . \cos x + \epsilon^x . \sin x = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{and } \partial_x^2 y + 2\epsilon^x . \sin x = 0 \ldots\ldots\ldots\ldots\ldots (3);$$

$\therefore$ by (1), (2), and (3) $\partial_x^2 y - 2 . \partial_x y + 2y = 0$.

90. If $u = 0$ be an equation between *three* variables, as x, y, z, z being considered a function of x and y, the three equations $u = 0$, $\partial_{(x)}u + \partial_{(z)}u.\partial_{(x)}z = 0$, $\partial_{(y)}u + \partial_{(z)}u.\partial_{(y)}z = 0$, hold together for the same values of the variables (Art. 75); therefore one constant or function occurring in any two of these equations may be eliminated between them; and two constants or functions may be eliminated between the three equations. In like manner by proceeding to higher orders of "partial differentials" a greater number of constants or functions may be eliminated.

Also, since, when $u = f(x, y, z) = 0$, by Art. 76,

$$\partial_{(x)}u.\partial x + \partial_{(y)}u.\partial y + \partial_{(z)}u.\partial z = 0,$$

a constant or function may be eliminated between these two equations.

Ex. $(x + y + z).\epsilon^z = c$; required to eliminate ϵ^z.

$$u = (x + y + z).\epsilon^z - c = 0$$

$$\partial_{(x)}u = \epsilon^z$$

$$\partial_{(y)}u = \epsilon^z$$

$$\partial_{(z)}u = \epsilon^z + (x + y + z)\,\epsilon^z;$$

$$\therefore\ \epsilon^z.(\partial x + \partial y + \partial z + \overline{x + y + z}.\partial z) = 0$$

and $\partial x + \partial y + \partial z + \overline{x + y + z}.\partial z = 0.$

91. The functions hitherto eliminated have been of known forms, but general or arbitrary functions also may be eliminated by the same methods.

Ex. 1. $z = \phi\sqrt{x^2 + y^2}$; required to eliminate $\phi\sqrt{x^2 + y^2}$.

Let $\sqrt{x^2 + y^2}$ be represented by r.

$$\text{Then } \partial_{(x)}z = \partial_{(r)}\phi r.\partial_{(x)}r$$

$$= \partial_{(r)}\phi r.\frac{x}{r},$$

$$\text{and } \partial_{(y)}z = \partial'_{(r)}\phi r.\frac{y}{r};$$

$$\therefore\ y.\partial_{(x)}z - x.\partial_{(y)}z = 0.$$

Ex. 2. $y - nz = F\overline{x - mz}$; to eliminate $F\overline{x - mz}$.

Let $x - mz$ be represented by r, then

$$y - nz - Fr = 0;$$

$$\therefore\ -\partial_{(r)}Fr - (n - m.\partial_{(r)}Fr).\partial_{(x)}z = 0\ldots\ldots(1),$$

$$\text{and } 1 - (n - m.\partial_{(r)}Fr).\partial_{(y)}z = 0\ldots\ldots(2);$$

$$\therefore\ \partial_{(r)}Fr.(1 - m.\partial_{(x)}z) = -n.\partial_{(x)}z,$$

$$\text{and } -\partial_{(r)}Fr.m.\partial_{(y)}z = 1 - n.\partial_{(y)}z;$$

$$\therefore\ \frac{1 - m.\partial_{(x)}z}{m.\partial_{(y)}z} = \frac{n.\partial_{(x)}z}{1 - n.\partial_{(y)}z},$$

$$\text{or } m.\partial_{(x)}z + n.\partial_{(y)}z = 0.$$

Ex. 3. $y = F\overline{x + at} + f\overline{x - at}$; required to eliminate both the general functions.

Let $x + at$ be represented by r, and $x - at$ by s,

$$\text{then } y = Fr + fs;$$

$$\therefore\ \partial_{(x)}y = \partial_{(r)}Fr.\partial_{(x)}r + \partial_{(r)}fs.\partial_{(x)}s$$

$$= \partial_{(r)}Fr + \partial_{(s)}fs.$$

$$\partial^2_x y = \partial^2_{(r)}Fr.\partial_{(x)}r + \partial^2_{(s)}fs.\partial_{(x)}s$$

$$= \partial^2_{(r)}Fr + \partial^2_{(s)}fs.$$

$$\partial_{(t)}y = \partial_{(r)}Fr.\partial_{(t)}r + \partial_{(s)}fs.\partial_{(t)}s$$

$$= a.\partial_{(r)}Fr - a.\partial_{(s)}fs.$$

$$\partial^2_{(t)}y = a^2.\partial^2_{(r)}Fr - a^2.\partial_{(s)}fs;$$

$$\therefore\ \partial^2_{(t)}y = a^2.\partial^2_{(x)}y.$$

92. If $u=0$ be an equation between x and y, involving n constants, a, b, c, &c., we may, by differentiating merely, or between the two equations $u=0$, $\partial_x u=0$, eliminate one of the constants, as a, and obtain what is called the "first derivative" of the "primitive" $u=0$. If b had been eliminated instead of a, a different "first derivative" would have been the result; and thus n different "first derivatives" may be obtained, each of which will contain $\overline{n-1}$ constants.

By again differentiating and eliminating, we may obtain "second derivatives" in which two constants will have disappeared, and the number of these will manifestly be equal to the number of combinations of n things taken two and two, that is, $n.\frac{n-1}{2}$.

Similarly, the number of "derivatives of the r^{th} order," in which r constants have disappeared, is

$$\frac{n.(n-1).(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}.$$

The number of "$\overline{n-1}|^{\text{th}}$ derivatives" in which only one constant remains is n; and, lastly, the number of "n^{th} derivatives," in which all the constants have disappeared, is 1.

Cor. A "derivative of the n^{th} order" has n "first primitives," $n.\frac{n-1}{2}$ "second primitives," and, lastly, one original "primitive" involving n arbitrary constants.

III. VALUES OF FUNCTIONS WHICH APPEAR UNDER THE FORM $\frac{0}{0}$.

93. Functions of one variable, as x, which are made to assume the form $\frac{0}{0}$ by giving some particular value, as a, to the variable, may frequently be put under the form $\frac{P.(x-a)^m}{Q.(x-a)^n}$, where m and n are both positive, and P and Q

do not become equal to 0 when a is substituted for x. In such cases the value of the function under the form $\frac{0}{0}$ may be obtained by dividing the numerator and denominator of $\frac{P.(x-a)^m}{Q.(x-a)^n}$, by $(x-a)^n$ or $(x-a)^m$, according as m is $>$ or $<n$, and then substituting a for x in the result.

Thus, if $m=n$, the result will be $\left\{\frac{P}{Q}\right\}_{x=a}$ [*].

If $m>n$, ... 0.

If $m<n$, ... ∞, infinitely great.

Ex. Required the value of $\left\{\frac{(x^2-a^2)^{\frac{3}{2}}}{(x^3-a^3)^{\frac{3}{2}}}\right\}_{x=a}$

$$\frac{(x^2-a^2)^{\frac{3}{2}}}{(x^3-a^3)^{\frac{3}{2}}}=\frac{(x+a)^{\frac{3}{2}}.(x-a)^{\frac{3}{2}}}{(x^2+ax+a^2)^{\frac{3}{2}}.(x-a)^{\frac{3}{2}}}$$

$$=\frac{(x+a)^{\frac{3}{2}}}{(x^2+ax+a^2)^{\frac{3}{2}}};$$

$\therefore$ the value required $=\left(\frac{2}{3a}\right)^{\frac{3}{2}}$.

94. When the reduction of the proposed function to the form $\frac{P.(x-a)^m}{Q.(x-a)^n}$ is either difficult or impracticable, the value of the function when a is substituted for x may frequently be more easily found by differentiation.

Let $\frac{fx}{\phi x}$ be the proposed function, then since fx and ϕx respectively become equal to 0 when a is substituted for x, they will also be equal to 0 when $x+a-x$, or $x+h$, is substituted for x, h being a quantity which becomes 0 when a is substituted for x; and, therefore,

---

[*] This notation signifies the value of $\frac{P}{Q}$ when a is substituted for x, or when $x=a$.

$$\frac{fx}{\phi x} = \frac{f\,\overline{x+h} - fx}{\phi\,\overline{x+h} - \phi x}$$

$$= \frac{(f\,\overline{x+h} - fx) \div h}{(\phi\,\overline{x+h} - \phi x) \div h}.$$

But the numerator and denominator of this latter fraction become respectively $\partial_x fx$, and $\partial_x \phi x$ when 0 is substituted for h, or x becomes a;

$$\therefore \left\{\frac{fx}{\phi x}\right\}_{x=a} = \left\{\frac{\partial_x fx}{\partial_x \phi x}\right\}_{x=a};$$

which latter function may not appear under the form $\frac{0}{0}$, and will in such case give the value required.

If $\left\{\frac{\partial_x fx}{\partial_x \phi x}\right\}_{x=a}$ should appear under the form $\frac{0}{0}$, the same process as before being repeated will give

$$\left\{\frac{\partial_x fx}{\partial_x \phi x}\right\}_{x=a} = \left\{\frac{\partial^2_x fx}{\partial^2_x \phi x}\right\}_{x=a};$$

$$\text{and } \therefore \left\{\frac{fx}{\phi x}\right\}_{x=a} = \left\{\frac{\partial^2_x fx}{\partial^2_x \phi x}\right\}_{x=a}; \text{ and so on,}$$

until a fraction be obtained which is not of the form $\frac{0}{0}$.

Cor. If $\frac{\partial^r_x fx}{\partial^r_x \phi x}$, the first of the fractions which does not become $\frac{0}{0}$ when a is substituted for x, is of the form $\frac{\infty}{\infty}$, the numerator and denominator must be respectively inverted, in which case the resulting fraction $\left\{\frac{\frac{1}{\partial^r_x fx}}{\frac{1}{\partial^r_x \phi x}}\right\}_{x=a}$ will be of the form $\frac{0}{0}$ and may be proceeded with as before.

Ex. 1. Required the value of $\left\{\dfrac{x^{n+1} - a^{n+1}}{n+1}\right\}_{n=-1}$.

Let N represent the numerator and D the denominator.

Then $\partial_n N = x^{n+1}.\log_\epsilon x - a^{n+1}.\log_\epsilon a$, $\therefore\ (\partial_n N)_{n=-1} = \log_\epsilon \dfrac{x}{a}$

$\partial_n D = 1$ and $(\partial_n D)_{n=-1} = 1$;

$$\therefore\ \left(\frac{N}{D}\right)_{n=-1} = \log_\epsilon \frac{x}{a}.$$

Ex. 2. Required the value of $\left\{\dfrac{a - \sqrt{a^2 - x^2}}{x^2}\right\}_{x=0}$.

$$\partial_x N = \frac{x}{\sqrt{a^2 - x^2}}; \qquad \therefore\ (\partial_x N)_{x=0} = 0,$$

$$\partial_x D = 2x; \qquad \therefore\ (\partial_x D)_{x=0} = 0,$$

$$\partial_x^2 N = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}},\ \text{(Art. 29)};\ \therefore\ (\partial_x^2 N)_{x=0} = \frac{1}{a},$$

$$\partial_x^2 D = 2; \qquad \therefore\ (\partial_x^2 D)_{x=0} = 2.$$

$$\therefore\ \left(\frac{N}{D}\right)_{x=0} = \frac{1}{2a}.$$

Ex. 3. Required the value of $\left\{\left(1 - \dfrac{2x}{\pi}\right).\tan x\right\}_{x=\frac{\pi}{2}}$.

The proposed function appears under the form $0 \times \infty$, but may be put under the form $\dfrac{0}{0}$ thus, $\left\{\dfrac{1 - \dfrac{2x}{\pi}}{\cot x}\right\}_{x=\frac{\pi}{2}}$.

In this case,

$$\partial_x N = -\frac{2}{\pi}; \qquad \therefore (\partial_x N)_{x=\frac{\pi}{2}} = -\frac{2}{\pi},$$

$$\partial_x D = -(\text{cosec}\,x)^2; \qquad \therefore (\partial_x D)_{x=\frac{\pi}{2}} = -1;$$

$$\therefore \text{ the value required is } \frac{-\frac{2}{\pi}}{-1} \text{ or } \frac{2}{\pi}.$$

95. If, in functions which can be put under the form $\frac{P.(x-a)^m}{Q.(x-a)^n}$, m and n are both fractional, it is clear that by the method of differentiation alone the factor which is common to the numerator and denominator can never be made to disappear, and therefore the resulting fractions $\left(\frac{\partial_x f x}{\partial_x \phi x}\right)_{x=a}$, $\left\{\frac{\partial_x^2 f x}{\partial_x^2 \phi x}\right\}_{x=a}$, &c. will always be of the form $\frac{0}{0}$. Hence in such cases the method by differentiation is inapplicable; and functions of this kind must either be actually reduced to the form $\frac{P.(x-a)^m}{Q.(x-a)^n}$ and have their values found, when a is substituted for x, as in Art. 92, or be treated in the manner pointed out in Wood's Algebra, Art. 376. 10th Ed.

96. If $\frac{f(x,y)}{\phi(x,y)}$ be a function which becomes $\frac{0}{0}$ when some particular values are substituted for x and y, $F(x,y) = 0$ being an equation by which the variables are connected, the value of $\frac{f(x,y)}{\phi(x,y)}$ in such case may also be determined by differentiation.

For, if the value of y in terms of x be obtained from the equation $F(x,y) = 0$, and substituted in $\frac{f(x,y)}{\phi(x,y)}$, the resulting fraction will then be a function of x only, and will fall within the method given in Art. 93. And therefore, since the *values* of $\partial_x f(x,y)$ and $\partial_x \phi(x,y)$ must be the same although y has not been eliminated before differentiation, the value of $\frac{f(x,y)}{\phi(x,y)}$ in the proposed case may be obtained by differentiating the numerator and denominator, considering y a function of x, substituting the particular values of x and y, and also substituting the corresponding value of $\partial_x y$ obtained from the equation $\partial_x F(x,y) = 0$.

Ex. Required the value of $\left\{\frac{y-a}{x}\right\}_{\substack{x=0 \\ y=a}}$, x and y being connected by the equation $y^2 - 2ax - a^2 = 0$.

$$\text{Here } \partial_x N = \partial_x y,$$

$$\partial_x D = 1,$$

$$\text{and } \partial_x y = \frac{a}{y};$$

$$\therefore \text{ the value required} = \left\{\frac{\frac{a}{y}}{1}\right\}_{y=a} = 1.$$

97. If $\partial_x y$, determined from the given equation $u = F(x,y) = 0$, coincide with the quantity itself, whose value under the form $\frac{0}{0}$ is required, it is clear that the above method may fail to give a definite result.

In such case, however, the value of the function, or of $\partial_x y$, may still be found.

For, since $-\frac{\partial_{(x)} u}{\partial_{(y)} u}$ is the value of $\partial_x y$ as obtained from the equation $u = 0$, (Art. 74.), it is identical with the proposed function which appears under the form $\frac{0}{0}$ for the particular values of x and y;

$$\therefore\ \partial_x y = -\frac{\partial_x(\partial_{(x)} u)}{\partial_x(\partial_{(y)} u)} \qquad \text{for those values,}$$

$$= -\frac{\partial^2_{(x)} u + \partial_{(y)}\partial_{(x)} u\,.\,\partial_x y}{\partial_{(x)}\partial_{(y)} u + \partial^2_{(y)} u\,.\,\partial_x y} \cdots\cdots\cdots\cdots\cdots\cdots (a)$$

$$\therefore\ \partial^2_{(x)} u + 2.\partial_{(x)}\partial_{(y)} u\,.\,\partial_x y + \partial^2_{(y)} u\,.\,(\partial_x y)^2 = 0$$

a quadratic equation for determining the required values of $\partial_x y$, or of the proposed function, when the particular values of x and y have been substituted in it.

If in the above equation the terms are severally equal to 0 independently of $\partial_x y$, the values of $\partial_x y$ cannot be determined from it. In such case, however, the fraction (a) would appear under the form $\frac{0}{0}$, if any constant quantity were substituted for $\partial_x y$. Hence, if $\partial_x y$ be treated as a constant,

$$\partial_x y = -\frac{\partial_x(\partial^2_{(x)} u + \partial_{(y)}\partial_{(x)} u\,.\,\partial_x y)}{\partial_x(\partial_{(x)}\partial_{(y)} u + \partial^2_{(y)} u\,.\,\partial_x y)} \quad \text{for the particular values of } x \text{ and } y,$$

$$= -\frac{\partial^3_{(x)} u + 2.\partial_{(y)}\partial^2_{(x)} u\,.\,\partial_x y + \partial_{(x)}\partial^2_{(y)} u\,.\,(\partial_x y)^2}{\partial^2_{(x)}\partial_{(y)} u + 2.\partial_{(x)}\partial^2_{(y)} u\,.\,\partial_x y + \partial^3_{(y)} u\,.\,(\partial_x y)^2} \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\therefore\ \partial^3_{(x)} u + 3.\partial^2_{(x)}\partial_{(y)} u\,.\,\partial_x y + 3.\partial_{(x)}\partial^2_{(y)} u\,.\,(\partial_x y)^2 + \partial^3_{(y)} u\,.\,(\partial_x y)^3 = 0$$

a cubic equation for determining the required values of $\partial_x y$, or of the proposed function, when the particular values of x and y have been substituted in it.

And the same process may, if necessary, be repeated, until an equation is obtained in which the terms are not severally equal to 0, when the particular values of x and y are substituted in it.

Obs. It is obvious, from the preceding Art., that $\partial_x y$ has *two* values at least for the particular values of x and y which make it appear under the form $\frac{0}{0}$; *three*, if the same values of x and y make the partial differentials of the second order, viz. $\partial^2_{(x)}u$, $\partial_{(x)}\partial_{(y)}u$, and $\partial^2_{(y)}u$, severally equal to 0; *four*, if the same values of x and y make the partial differentials of the third order severally equal to 0; and so on.

Ex. $u = x^4 - ayx^2 + by^3 = 0$. Required the values of $(\partial_x y)_{\substack{x=0\\y=0}}$.

$$\left.\begin{aligned}\partial_{(x)}u &= 4x^3 - 2ayx\\ \partial_{(y)}u &= 3by^2 - ax^2\end{aligned}\right\};\ \therefore\ (\partial_x y)_{\substack{x=0\\y=0}} = \frac{0}{0}.$$

$$\left.\begin{aligned}\partial^2_{(x)}u &= 12x^2 - 2ay\\ \partial_{(x)}\partial_{(y)}u &= -2ax\\ \partial^2_{(y)}u &= 6by\end{aligned}\right\}\ \begin{array}{c}\text{which are severally equal to 0,}\\ \text{when } x = 0, \text{ and } y = 0.\end{array}$$

$$\left.\begin{aligned}\partial^3_{(x)}u &= 24x\\ \partial^2_{(x)}\partial_{(y)}u &= -2a\\ \partial_{(x)}\partial^2_{(y)}u &= 0\\ \partial^3_{(y)}u &= 6b\end{aligned}\right\}\ \begin{array}{c}\text{some of which remain definite quan-}\\ \text{tities, when } x = 0, \text{ and } y = 0.\end{array}$$

Therefore substituting these quantities in the equation

$$\partial^3_{(x)}u + 3.\partial_{(y)}\partial^2_{(x)}u.\partial_x y + 3.\partial_{(x)}\partial^2_{(y)}u.(\partial_x y)^2 + \partial^3_{(y)}u.(\partial_x y)^3 = 0;$$

$$-6a.\partial_x y + 6b.(\partial_x y)^3 = 0,$$

is an equation for determining the required values of $\partial_x y$;

$$\therefore\ (\partial_x y)_{\substack{x=0\\y=0}} = 0, \text{ and } \pm\sqrt{\frac{a}{b}}.$$

SECTION IX.

EXPANSIONS.

98. **Lemma. I.** *If $\phi(x,h)$ be a function of x and h which becomes equal to the definite function ϕx when 0 is substituted for h, then, when h is diminished indefinitely, the difference between $h^n.\phi(x,h)$ and $h^n.\phi x$ may be made so small that it may be neglected when compared with either of them.*

For $h^n.\phi(x,h) \sim h^n.\phi x : h^n.\phi x = \phi(x,h) \sim \phi x : \phi x$.

But it is evident that, by diminishing h, the latter ratio and therefore also the former, may be diminished indefinitely. Thus it appears that $h^n.\{\phi(x,h) \sim \phi x\}$ may be neglected when compared with $h^n.\phi x$; and similarly, when compared with $h^n.\phi(x,h)$.

Cor. Hence, in any calculations or equations in which $h^n.\phi(x,h)$ occurs, when h is diminished indefinitely $h^n.\phi x$ may be substituted for $h^n.\phi(x,h)$.

Lemma. II. *If a function can be put under the form*

$$A_0 + A_1.h + A_2.h^2 + \ldots\ldots\ldots A_{n-1}.h^{n-1} + P_n.h^n$$

A_0, A_1, A_2, &c. being definite functions of x only, and P_n a function of x and h which becomes equal to A_n when 0 is substituted for h, the series $A_0 + A_1.h + A_2.h^2 + \ldots\ldots + A_n.h^n$ is an approximation to the value of the function when h is diminished indefinitely, and the approximation will be the nearer, the greater n is.

For, by Lemma I, when h is diminished indefinitely, the difference between $P_n.h^n$ and $A_n.h^n$ may be neglected when compared with either of them, and $A_n.h^n$ may be substituted for $P_n.h^n$; therefore the latter series is an approximation to the function.

Again, since $P_n.h^n$ and $P_{n+1}.h^{n+1}$ respectively express the remainder after n and $\overline{n+1}$ terms, and since

$$P_n.h^n = A_n.h^n + P_{n+1}.h^{n+1};$$

$$\therefore \frac{P_{n+1}.h^{n+1}}{P_n.h^n} = \frac{P_n - A_n}{P_n}.$$

But, by diminishing h, $P_n - A_n$ may be made less than P_n, and therefore $P_{n+1}.h^{n+1}$ less than $P_n.h^n$ for such value of h. Hence, by sufficiently diminishing h, the quantity neglected in taking only n terms of the series is less the greater n is.

COR. It has been proved (Lem. I.) that the difference between $A_n.h^n$ and $P_n.h^n$, when h is diminished indefinitely, may be neglected compared with either of them; but this difference is the remainder of the series after $A_n.h^n$:—therefore, by sufficiently diminishing h, any one term may be made greater than the sum of all the succeeding terms.

99. *If the number of differentials with regard to x of fx be limited, so that all of a higher order than $\partial_x^n fx$ are separately equal to* 0, *then*

$$fx + \partial_x fx.h + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{h^n}{\underline{|n}} \text{ is equal to } f\overline{x+h},$$

whatever be the value of h. And if the number of such successive differentials be unlimited, the above series is an approximation to the value of $f\overline{x+h}$, when h is diminished indefinitely; and the approximation will be the nearer the greater n is, that is, the greater the number of terms of the series which are taken. [TAYLOR'S THEOREM.]

For, if $\dfrac{fz - fx}{z - x}$ be represented by $\phi(z, x)$, it is clear that

$$fz = fx + \phi(z, x).\{z - x\}, \text{ whatever } x \text{ be};$$

therefore let $x + h$ be substituted for x, then

$$fz = f\overline{x+h} + \phi(z, \overline{x+h}).\{z - x - h\};$$

$\therefore$ subtracting and dividing by h,

$$0=\frac{f\overline{x+h}-fx}{h}+\frac{\phi(z,\overline{x+h})-\phi(z,x)}{h}\{z-x\}-\phi(z,\overline{x+h}),$$

and substituting 0 for h, the first term becomes a definite function $\partial_x fx$; $\phi(z,\overline{x+h})$ becomes a definite function $\phi(z,x)$; therefore the remaining term of the equation must also become a definite function, which will consequently be $\partial_x\phi(z,x).\{z-x\}$. Hence the equation becomes

$$0=\partial_x fx+\partial_x\phi(z,x).\{z-x\}-\phi(z,x)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (\text{I}).$$

Again, by substituting $x+h$ for x in this last equation and proceeding as before,

$$0=\partial_x^2 fx+\partial_x^2\phi(z,x).\{z-x\}-2.\partial_x\phi(z,x)\ \ldots\ldots\ldots\ldots\ (\text{II}).$$

Similarly

$$0=\partial_x^3 fx+\partial_x^3\phi(z,x).\{z-x\}-3.\partial_x^2\phi(z,x)\ldots\ldots\ldots\ldots\ (\text{III}).$$

And generally

$$0=\partial_x^{n-1}fx+\partial_x^{n-1}\phi(z,x).\{z-x\}-(n-1).\partial_x^{n-2}\phi(z,x)\ldots(N-1);$$

and by successively substituting in the original equation the values of $\phi(z,x)$, $\partial_x\phi(z,x)$, &c. deduced from the equations (I), (II), (III) &c.

$$fz=fx+\partial_x fx.\frac{z-x}{1}+\partial_x^2 fx.\frac{z-x}{\underline{|2}}+\ldots+\partial_x^{n-1}fx.\frac{(z-x)^{n-1}}{\underline{|n-1}}$$

$$+\partial_x^{n-1}\phi(z,x).\frac{(z-x)^n}{\underline{|n-1}}\ldots(\text{A}).$$

(1) Now, if the number of the successive differentials of fx be limited to n, so that $\partial_x^{n+1}fx=0$, substituting $\overline{n+1}$ for $n-1$ in equation $(N-1)$

$$0=\partial_x^{n+1}fx+\partial_x^{n+1}\phi(z,x).\{z-x\}-\overline{n+1}.\partial_x^n\phi(z,x);$$

$$\therefore\ 0=\partial_x^{n+1}\phi(z,x).\{z-x\}-\overline{n+1}.\partial_x^n\phi(z,x);$$

$$\therefore\ \frac{\partial_x^{n+1}\phi(z,x)}{\partial_x^n\phi(z,x)}=\frac{n+1}{z-x}.$$

$$\text{or}\quad \frac{\partial_x^{n+1}\phi(z,x)}{\partial_x^{n}\phi(z,x)} + (n+1).\frac{\partial_x(z-x)}{z-x} = 0.$$

And since $\dfrac{\partial_x u}{u} = \partial_x \log_e u$, (Art. 32.), the above equation would have been obtained by differentiating the equation

$$\log_e \partial_x^n \phi(z,x) + (n+1).\log_e \overline{z-x} = \log_e c;$$

$$\therefore\ \partial_x^n \phi(z,x).\{z-x\}^{n+1} = c.$$

Hence, taking one more term of the series (A),

$$fz = fx + \partial_x fx.\frac{z-x}{1} + \partial_x^2 fx.\frac{z-x}{1\times 2} + \ldots\ldots + \partial_x^n fx.\frac{z-x}{\underline{|n}} + \frac{c}{\underline{|n}},$$

an equation which holds for all values whatever of x; but when $x=z$, $c=0$, hence c is always equal to 0, and therefore

$$fz = fx + \partial_x fx.\frac{z-x}{1} + \partial_x^2 fx.\frac{z-x}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{(z-x)^n}{\underline{|n}}.$$

(2) If the number of differentials $\partial_x fx$, $\partial_x^2 fx$, &c. be unlimited:—since

$$\phi(z,x) = \frac{fz - fx}{z-x},$$

$$\partial_x \phi(z,x) = \frac{\phi(z,x) - \partial_x fx}{z-x}, \quad \text{from equation} \quad \text{(I)}$$

$$\partial_x^2 \phi(z,x) = \frac{2.\partial_x \phi(z,x) - \partial_x^2 fx}{z-x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(II)}$$

$$\partial_x^3 \phi(z,x) = \frac{3.\partial_x^2 \phi(z,x) - \partial_x^3 fx}{z-x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(III)}$$

$$\&\text{c.} = \&\text{c.}\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore\ \{\phi(z,x)\}_{x=z} = \frac{0}{0} = \left\{\frac{-\partial_x fx}{-1}\right\}_{x=z} = (\partial_x fx)_{x=z}.$$

$$\text{Also}\ \{\partial_x \phi(z,x)\}_{x=z} = \frac{0}{0} = \left\{\frac{\partial_x \phi(z,x) - \partial_x^2 fx}{-1}\right\}_{x=z};$$

$$\therefore\ 2.\{\partial_x \phi(z,x)\}_{x=z} = (\partial_x^2 fx)_{x=z}.$$

Similarly, $3.\{\partial_z^2\phi(z,x)\}_{z=x} = (\partial_x^3 fx)_{z=x}$.

And generally $n.\{\partial_z^{n-1}\phi(z,x)\}_{z=x} = (\partial_x^n fx)_{z=x}$;

$$\therefore \frac{1}{\underline{|n-1}}.\{\partial_z^{n-1}\phi(z,x)\}_{z=x} = (\partial_x^n fx)_{z=x}.\frac{1}{\underline{|n}},$$

which is the value of the coefficient of $(z-x)^n$ in the series (A) when $x=z$.

Hence, by Lemma II, (since $z-x$ corresponds to h,) when $z-x$ is diminished indefinitely,

$$fx + \partial_x fx.\frac{z-x}{1} + \partial_x^2 fx.\frac{(z-x)^2}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{(z-x)^n}{\underline{|n}}$$

is an approximation to the value of fz, and the approximation is the nearer, the greater n is.

100. In both the cases discussed in the preceding Art., since fx, $\partial_x fx$, $\partial_x^2 fx$, &c. do not contain z, these quantities will not be altered, whatever be substituted for z; therefore, substituting $x+h$ for z,

$$fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{h^n}{\underline{|n}}$$

is equal to $f\overline{x+h}$, for any value of h, if the number of successive differentials of fx is limited to n; and is an approximation to $f\overline{x+h}$, when h is diminished indefinitely and the number of differentials is unlimited, which may be made the nearer, the greater n is taken.

Cor. It is clear that, (as in Cor. Art. 98,) by sufficiently diminishing h, any one term of the above series may be made greater than the sum of all the succeeding terms. Hence, the quantity neglected by taking only a certain number of terms reckoned from the beginning of the series is always less than the last of the terms taken.

101. The only functions of x, whose successive differentials are known to be limited in number to n, may be put under the form $Ax^n + Bx^p + Cx^q +$ &c. n, p, q, &c. being positive integers arranged in descending order of magnitude, and A, B, C, &c. quantities which do not contain x.

For functions of this kind, the equation

$$fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{h^n}{\underline{|n}} = f\overline{x+h}$$

may be more readily proved by means of the Binomial Theorem.

Thus, if $fx = A.x^n + B.x^p + C.x^q +$ &c.

$$f\overline{x+h} = A(x+h)^n + B(x+h)^p + C(x+h)^q + \text{\&c.}$$

$$= Ax^n + Bx^p + Cx^q + \text{\&c.}$$

$$+ (nAx^{n-1} + pBx^{p-1} + qCx^{q-1} + \text{\&c.}).h$$

$$+ \{n(n-1)Ax^{n-2} + p(p-1)Bx^{p-2} + q(q-1)Cx^{q-2} + \text{\&c.}\}\frac{h^2}{\underline{|2}}$$

$$+ \quad \text{\&c.} \qquad \text{\&c.} \qquad + Ah^n$$

by the Binomial Theorem, which is strictly proved for positive integral powers of $\overline{x+h}$, (*Wood's* Algebra, Art. 232).

But since

$$fx = Ax^n + Bx^p + Cx^q + \text{\&c.}$$

$$\partial_x fx = nAx^{n-1} + pBx^{p-1} + qCx^{q-1} + \text{\&c.}$$

$$\partial_x^2 fx = n(n-1).Ac^{n-2} + p(p-1)Bx^{p-2} + q(q-1).Cx^{q-2} + \text{\&c.}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\partial_x^n fx = \underline{|n}.A\,;$$

$$\therefore\ f\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \ldots\ldots + \partial_x^n fx.\frac{h^n}{\underline{|n}}.$$

102. *If $f\overline{x+h}$ can be expanded by any method in powers of h, one or more of these powers being fractional, and the series be arranged in ascending order of magnitude of the exponents of h, Taylor's series will correctly represent the* FORM *of the expansion as far as the first fractional power, but no farther.*

Let the series be represented by

$$A_0 + A_1.h + A_2.h^2 + \ldots\ldots + A_n.h^n + B.h^m + \&c.$$

any of which quantities A_0, A_1, A_2, &c. may be equal to 0, and the index m being between n and $n+1$.

Then, since $\partial_{x+h} f\overline{x+h} = \partial_h f\overline{x+h}$,

$$\begin{array}{ll} \text{and } f\overline{x+h} = & A_0 + A_1.h + A_2.h^2 + \ldots\ldots + A_n.h^n + B.h^m + \&c. \\ \partial_{x+h} f\overline{x+h} = & A_1 + 2A_2.h + \ldots\ldots + nA_n.h^{n-1} + mB.h^{m-1} + \&c. \\ \partial^2_{x+h} f\overline{x+h} = & 2A_2 + \ldots n(n-1).A_n.h^{n-2} + m(m-1).Bh^{n-2} + \&c. \\ \ldots\ldots\ldots\ldots = & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \partial^n_{x+h} f\overline{x+h} = & \quad \lfloor\!\underline{n}.A_n + m(m-1)\ldots(m-n+1).Bh^{m-n} + \&c. \end{array}$$

Hence, when 0 is substituted for h,

$$\begin{aligned} fx &= A_0 \\ \partial_x fx &= A_1 \\ \partial^2_x fx &= A_2 \\ \&c. &= \&c. \\ \partial^n_x fx &= A_n. \end{aligned}$$

Also, since $\partial^{n+1}_{x+h} f\overline{x+h} = m(m-1)\ldots(m-n)Bh^{m-(n+1)} + \&c.$ and m is less than $n+1$, this latter quantity becomes infinite when 0 is substituted for h. Hence $\partial^{n+1}_x fx = \infty$, and the same is evidently true of every succeeding differential of fx.

COR. If the expansion of $f\overline{x+h}$ contain a negative power of h, it is evident that fx, and its successive differentials, are respectively infinite.

103. If the original function fx, or any one of its differentials, as $\partial_x^r fx$, is infinite, then $\frac{1}{fx} = 0$, or $\frac{1}{\partial_x^r fx} = 0$, and is therefore an equation for determining x. Hence, conversely, this can only happen when a particular value is given to x; and therefore sometimes, when a particular value is given to x, the series in Art. 100 will not at all represent the *value* of $f\overline{x+h}$, although the coefficients of positive integral powers of h in its expansion will coincide with the corresponding coefficients in that series.

Ex. 1. Required to expand $(x+h)^5$.

$$\begin{aligned}
fx &= x^5,\\
\partial_x fx &= 5x^4,\\
\partial_x^2 fx &= 4.5.x^3,\\
\partial_x^3 fx &= 3.4.5.x^2,\\
\partial_x^4 fx &= 2.3.4.5.x,\\
\partial_x^5 fx &= 1.2.3.4.5;
\end{aligned}$$

∴ substituting in the Theorem

$$f\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \ldots\ldots + \partial_x^5 fx.\frac{h^5}{\underline{|5}},$$

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5.$$

Ex. 2. Required to expand $\sin\overline{x+h}$, and $\cos\overline{x+h}$, in series of ascending integral and positive powers of h.

$$\begin{aligned}
fx &= \sin x,\\
\partial_x fx &= \cos x,\\
\partial_x^2 fx &= -\sin x,\\
\partial_x^3 fx &= -\cos x,\\
\partial_x^4 fx &= \sin x,\\
\&c. &= \&c.;
\end{aligned}$$

∴ by the Theorem,

$$\sin\overline{x+h} = \sin x + \cos x.\frac{h}{1} - \sin x.\frac{h^2}{\lfloor\underline{2}} - \cos x.\frac{h^3}{\lfloor\underline{3}} + \ldots\ldots$$

Similarly,

$$\cos\overline{x+h} = \cos x - \sin x.\frac{h}{1} - \cos x.\frac{h^2}{\lfloor\underline{2}} + \sin x.\frac{h^3}{\lfloor\underline{3}} + \ldots\ldots$$

Ex. 3. Required to expand $\log\overline{x+h}$.

$$fx = \log_e x;$$

$$\therefore\ \partial_x fx = \frac{1}{x},.$$

$$\partial_x^2 fx = -\frac{1}{x^2},$$

$$\partial_x^3 fx = \frac{1.2}{x^3},$$

and generally $\partial_x^n fx = \dfrac{\lfloor\underline{n-1}}{x^n}$;

$$\therefore\ \log_e\overline{x+h} = \log_e x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \&c. + \frac{h^n}{n.x^n}.(-1)^{n+1} + \ldots$$

104. *If a function of one variable can be expanded in a series of ascending integral and positive powers of the variable, to find the form of the expansion.* [*MACLAURIN'S THEOREM.*]

Let u represent the function capable of being expanded in a series ascending by positive and integral powers of x, and assume

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$$

Then $\partial_x u = a_1 + 2a_2 x + 3a_3 x^2 + \&c.$

$\partial_x^2 u = 1.2a_2 + 2.3a_3 x + \&c.$

$\partial_x^3 u = 1.2.3\,a_3 + \&c.$;

$$\therefore\ u_{x=0} = a_0,$$
$$(\partial_x u)_{x=0} = a_1,$$
$$(\partial_x^2 u)_{x=0} = 1.2.a_2,$$
$$(\partial_x^3 u)_{x=0} = 1.2.3.a_3,$$

and generally, $(\partial_x^n u)_{x=0} = \underline{|n}.a_n$.

Therefore, substituting in the assumed series the values of a_0, a_1, a_2, &c. deduced from these equations,

$$u = u_{x=0} + (\partial_x u)_{x=0}.\frac{x}{1} + (\partial_x^2 u)_{x=0}.\frac{x^2}{\underline{|2}} + (\partial_x^3 u)_{x=0}.\frac{x^3}{\underline{|3}} + \&c.$$

Obs. Since, in the preceding proof of Maclaurin's Theorem, it has been *assumed*, that u is capable of expansion in a series of positive integral powers of x, the mere fact of any number of the quantities $u_{x=0}$, $(\partial_x u)_{x=0}$, $(\partial_x^2 u)_{x=0}$, &c. being finite affords no proof that the function u can be so expanded.

If the function can be expanded in powers of x, one or more of the exponents being fractional, Maclaurin's Theorem will correctly represent the *form* of the expansion up to the first fractional power; but it is evident that it will not serve to determine the coefficients beyond that term, since the succeeding differentials of u will severally become infinite, when 0 is substituted for x.

105. *Any function whatever of x may be expanded by Maclaurin's Theorem, when x is indefinitely diminished, provided that neither the function nor any of its successive differentials with regard to x become infinite when* 0 *is substituted for x.*

For, by Art. 100, *generally*, when h is indefinitely diminished,

$$f\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{|2}} + \&c.\dots\dots$$

$$\therefore fh = (fx)_{x=0} + (\partial_x fx)_{x=0}.\frac{h}{1} + (\partial_x^2 fx)_{x=0}.\frac{h^2}{\lfloor 2} + \&c.$$

$$\text{or } fx = (fx)_{x=0} + (\partial_x fx)_{x=0}.\frac{x}{1} + (\partial_x^2 fx)_{x=0}.\frac{x^2}{\lfloor 2} + \&c.$$

when x is indefinitely diminished; since $(fx)_{x=0}$, $(\partial_x fx)_{x=0}$, &c. which do not contain either x or h, remain unaltered when x is substituted for h.

Ex. 1. Required to expand $(x+a)^5$.

$$u = (x+a)^5; \qquad \therefore \quad u_{x=0} = a^5.$$

$$\partial_x u = 5.(x+a)^4; \qquad \therefore \quad (\partial_x u)_{x=0} = 5a^4.$$

$$\partial_x^2 u = 4.5.(x+a)^3; \qquad \therefore \quad (\partial_x^2 u)_{x=0} = 4.5.a^3.$$

$$\partial_x^3 u = 3.4.5.(x+a)^2; \qquad \therefore \quad (\partial_x^3 u)_{x=0} = 3.4.5.a^2.$$

$$\partial_x^4 u = 2.3.4.5.(x+a); \qquad \therefore \quad (\partial_x^4 u)_{x=0} = 2.3.4.5.a.$$

$$\partial_x^5 u = 1.2.3.4.5; \qquad \therefore \quad (\partial_x^5 u)_{x=0} = 1.2.3.4.5.$$

Hence substituting in the Theorem

$$u = u_{x=0} + (\partial_x u)_{x=0}.\frac{x}{1} + (\partial_x^2 u)_{x=0}.\frac{x^2}{\lfloor 2} + \&c.$$

$$(x+a)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

Ex. 2. Required to expand a^x and ϵ^x.

$$u = a^x; \qquad \therefore \quad u_{x=0} = 1.$$

$$\partial_x u = \log_e a . a^x; \qquad \therefore \quad (\partial_x u)_{x=0} = \log_e a.$$

$$\partial_x^2 u = (\log_e a)^2.a^x; \qquad \therefore \quad (\partial_x^2 u)_{x=0} = (\log_e a)^2.$$

$$\partial_x^3 u = (\log_e a)^3.a^x; \qquad \therefore \quad (\partial_x^3 u)_{x=0} = (\log_e a)^3.$$

And generally

$$\partial_x^n u = (\log_\epsilon a)^n.a^x; \qquad \therefore \quad (\partial_x^n u)_{x=0} = (\log_\epsilon a)^n.$$

$$\therefore\ a^x = 1 + \log_\epsilon a.\frac{x}{1} + (\log_\epsilon a)^2.\frac{x^2}{\underline{|2}} + \ldots\ldots + (\log_\epsilon a)^n.\frac{x^n}{\underline{|n}} + \ldots (1).$$

Hence also $\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots + \frac{x^n}{\underline{|n}} + \ldots\ldots\ldots\ldots (2).$

NOTE. Since $a^x = \{1+(a-1)\}^x$

$=1+x(a-1)+x.\frac{x-1}{2}(a-1)^2+$ &c. (by the Bin. Theor. which is proved strictly for positive integral values of x)

$$=1+\left\{(a-1)-\frac{1}{2}(a-1)^2+\&c.\right\}x+\left\{\frac{1}{2}(a-1)^2+\&c.\right\}x^2+\&c.$$

i. e., since a^x is shewn to be capable of expansion in a series of ascending positive integral powers of x, independently of Maclaurin's Theorem, the above series (1) and (2) properly represent the *values* of a^x and ϵ^x for positive integral values of x. Hence, to find the numerical value of ϵ, let $x = 1$ in series (2), then

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots$$

$$= 2.7182818\ldots\ldots$$

This value of ϵ is the base of that system of logarithms which is usually called Napierian or hyperbolic; and is the value of $\left\{\frac{\epsilon^h - 1}{h}\right\}_{h=0}$; vide Art. 31.

Ex. 3. Required to expand $\sin x$ and $\cos x$ in powers of x.

$$\begin{array}{rll} u = & \sin x; & \therefore\ u_{x=0} = 0. \\ \partial_x u = & \cos x; & \therefore\ (\partial_x u)_{x=0} = 1. \\ \partial_x^2 u = & -\sin x; & \therefore\ (\partial_x^2 u)_{x=0} = 0. \\ \partial_x^3 u = & -\cos x; & \therefore\ (\partial_x^3 u)_{x=0} = -1. \\ \partial_x^4 u = & \sin x; & \therefore\ (\partial_x^4 u)_{x=0} = 0. \end{array}$$

From which it is evident that the coefficients of M'Laurin's series in this case are 0, 1, 0, -1, repeated continually in the same order;

$$\therefore\ \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots + (-1)^{n+1}.\frac{x^{2n-1}}{\underline{|2n-1}} + \ldots\ldots\ldots(1).$$

Similarly,

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots + (-1)^{n+1}.\frac{x^{2(n-1)}}{\underline{|2(n-1)}} + \ldots\ldots(2).$$

Ex. 4. Required to expand y in a series of ascending powers of x, y being an implicit function of x, given by the equation $x = ay + by^2 + cy^3 + \&c.$

$$\begin{aligned} x &= ay + by^2 + cy^3 + \&c. \\ \partial_y x &= a + 2by + 3cy^2 + \&c. \\ \partial_y^2 x &= 2b + 2.3cy + \&c. \\ \partial_y^3 x &= 1.2.3\,c + \&c. \\ \&c. &= \&c. \end{aligned}$$

$$\therefore\ y_{x=0} = 0,$$

$$(\partial_x y)_{x=0} = \frac{1}{(\partial_y x)_{x=0}} = \frac{1}{a},\ \text{Art. 87.}$$

$$(\partial_x^2 y)_{x=0} = \left\{\frac{-\partial_y^2 x}{(\partial_y x)^3}\right\}_{x=0} = -\frac{2b}{a^3},$$

$$(\partial_x^3 y)_{x=0} = \left\{\frac{3.(\partial_y^2 x)^2 - \partial_y^3 x\,.\,\partial_y x}{(\partial_y x)^5}\right\}_{x=0}$$

$$= \frac{1}{a^5}(3.4b^2 - 1.2.3ac),$$

$$\&c. = \&c.$$

$$\therefore\ y = \frac{x}{a} - \frac{b}{a^3}.x^2 + \frac{2b^2 - ac}{a^5}.x^3 + \&c.$$

106. Maclaurin's Theorem is applicable to the expansion of functions of one variable in *negative* integral powers of the variable, if they admit of such expansion.

For, let fx represent the function; then, if $z = \frac{1}{x}$, by the hypothesis $f\frac{1}{z}$ is capable of being expanded in positive integral powers of z.

$$\therefore f\frac{1}{z} = \left(f\frac{1}{z}\right)_{z=0} + \left(\partial_z f\frac{1}{z}\right)_{z=0} \cdot \frac{z}{1} + \left(\partial_z^2 f\frac{1}{z}\right)_{z=0} \cdot \frac{z^2}{\underline{|2}} + \&c.$$

But $\left(f\frac{1}{z}\right)_{z=0}$, $\left(\partial_z f\frac{1}{z}\right)_{z=0}$, $\left(\partial_z^2 f\frac{1}{z}\right)_{z=0}$, &c., are evidently equal to $\left(f\frac{1}{x}\right)_{x=0}$, $\left(\partial_x f\frac{1}{x}\right)_{x=0}$, $\left(\partial_x^2 f\frac{1}{x}\right)_{x=0}$, &c. respectively, and $z = x^{-1}$;

$$\therefore fx = \left(f\frac{1}{x}\right)_{x=0} + \left(\partial_x f\frac{1}{x}\right)_{x=0} \cdot \frac{x^{-1}}{1} + \left(\partial_x^2 f\frac{1}{x}\right)_{x=0} \cdot \frac{x^{-2}}{\underline{|2}} + \&c.$$

Ex. To expand $\left(a + \frac{b}{x} + \frac{c}{x^2}\right)^n$ in negative integral powers of x.

If $fx = \left(a + \frac{b}{x} + \frac{c}{x^2}\right)^n$,

$$f\frac{1}{x} = (a+bx+cx^2)^n; \quad \therefore \left(f\frac{1}{x}\right)_{x=0} = a^n.$$

$$\partial_x f\frac{1}{x} = n(a+bx+cx^2)^{n-1}.(b+2cx);$$

$$\therefore \left(\partial_x f\frac{1}{x}\right)_{x=0} = na^{n-1}b.$$

$$\partial_x^2 f\frac{1}{x} = n(n-1)\,(a+bx+cx^2)^{n-2}.(b+2cx)^2$$
$$+ n(a+bx+cx^2)^{n-1}.2c;$$

&c. = &c.

$$\therefore \left(\partial_x^2 f\frac{1}{x}\right)_{x=0} = n(n-1)a^{n-2}b^2 + 2na^{n-1}c.$$

&c. = &c.

$$\therefore\ \left(a+\frac{b}{x}+\frac{c}{x^2}\right)^n = a^n + na^{n-1}b.\frac{1}{x}$$

$$+ \left\{n(n-1)a^{n-2}b^2 + 2na^{n-1}c\right\}.\frac{1}{2x^2} + \&c.$$

the coefficient of $\frac{1}{x^r}$ being the value of $\left\{\partial^r_x.(a+bx+cx^2)^n\right\}_{x=0}.\frac{1}{\underline{|r}}$.

107. *To expand a function of* $\overline{x+h}$ *and* $\overline{y+k}$ *in a series of ascending positive and integral powers of h and k**.

Let $f(\overline{x+h},\ \overline{y+k})$ represent the proposed function,

and $\qquad u \qquad \ldots\ldots \qquad f(x,y)$;

then, if y be treated as the only variable, and $y+k$ substituted for it in $f(x,y)$, or u, by Taylor's Theorem

$$f(x,\overline{y+k}) = u + \partial_{(y)}u.\frac{k}{1} + \partial^2_{(y)}u.\frac{k^2}{\underline{|2}} + \partial^3_{(y)}u.\frac{k^3}{\underline{|3}} + \ldots\ldots$$

an equation which holds true whatever x be. But, if $x+h$ be substituted for x,

$$\begin{array}{lll}
f(x,\overline{y+k}) & \text{becomes} & f(\overline{x+h},\ \overline{y+k}) \\
u & \ldots\ldots & u + \partial_{(x)}u.\frac{h}{1} + \partial^2_{(x)}u.\frac{h^2}{\underline{|2}} + \partial^3_{(x)}u.\frac{h^3}{\underline{|3}} + \ldots \\
\partial_{(y)}u & \ldots\ldots & \partial_{(y)}u + \partial_{(x)}\partial_{(y)}u.\frac{h}{1} + \partial^2_{(x)}\partial_{(y)}u.\frac{h^2}{\underline{|2}} + \ldots \\
\partial^2_{(y)}u & \ldots\ldots & \partial^2_{(y)}u + \partial_{(x)}\partial^2_{(y)}u.\frac{h}{1} + \ldots \\
\partial^3_{(y)}u & \ldots\ldots & \partial^3_{(y)}u + \ldots
\end{array}$$

…………………………………………………………………

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* In this and the succeeding Arts., to the end of the Section, the brackets by which "*partial*" are distinguished from "*total*" differentials may, for the sake of convenience, be omitted, provided it be carefully borne in mind that none but "*partial*" differentials occur in any of them.

Hence, by substitution,

$$
\begin{aligned}
f(\overline{x+h}, \overline{y+k}) = u &+ \partial_{(x)} u.\frac{h}{1} + \partial_{(x)}^{2} u.\frac{h^2}{\underline{|2}} + \partial_{(x)}^{3} u.\frac{h^3}{\underline{|3}} + \ldots \\
&+ \partial_{(y)} u.\frac{k}{1} + \partial_{(x)}\partial_{(y)} u.\frac{h}{1}.\frac{k}{1} + \partial_{(x)}^{2}\partial_{(y)} u.\frac{h^2}{\underline{|2}}.\frac{k}{1} + \ldots \\
&+ \partial_{(y)}^{2} u.\frac{k^2}{\underline{|2}} + \partial_{(x)}\partial_{(y)}^{2} u.\frac{h}{1}.\frac{k^2}{\underline{|2}} + \ldots \\
&+ \partial_{(y)}^{3} u.\frac{k^3}{\underline{|3}} + \ldots \\
&+ \ldots
\end{aligned}
$$

This method of obtaining the expansion may evidently be extended to functions of any greater number of variables.

Ex. To expand $(x+h)^m.(y+k)^n$ in a series of ascending positive and integral powers of h and k.

$$
\begin{aligned}
\text{Here } u &= x^m y^n ; \\
\therefore\ \partial_{(x)} u &= m x^{m-1}.y^n, \\
\partial_{(x)}^{2} u &= m(m-1).x^{m-2}.y^n, \\
\partial_{(y)} u &= n x^m.y^{n-1}, \\
\partial_{(y)}^{2} u &= n(n-1)\, x^m.y^{n-2}, \\
\partial_{(x)}\partial_{(y)} u &= m n x^{m-1}.y^{n-1}, \\
\&\text{c.} &= \&\text{c.}
\end{aligned}
$$

Therefore, by the general Theorem,

$$
\begin{aligned}
(x+h)^m.(y+k)^n = x^m y^n &+ m x^{m-1}.y^n.\frac{h}{1} + m(m-1)\, x^{m-2}.y^n.\frac{h^2}{\underline{|2}} + \ldots \\
&+ n x^m.y^{n-1}.\frac{k}{1} + m n x^{m-1}.y^{n-1}.\frac{h k}{1} + \ldots \\
&+ n(n-1)\, x^m.y^{n-2}.\frac{k^2}{\underline{|2}} + \ldots \\
&+ \ldots
\end{aligned}
$$

108. From the expansion in the last Art. may be deduced a remarkable property of homogeneous functions, i. e. of functions in which the sum of the exponents of the variables in every term is the same.

Let u be a homogeneous function of x, y, z, &c. of m dimensions, the sum of the exponents of x, y, z, &c. in each term being m; and for x, y, z, &c. substitute $(1+h)x$, $(1+h)y$, $(1+h)z$, &c. Then it is evident that the function u will be the same as before, except that each term will be multiplied by $(1+h)^m$, or u will become $(1+h)^m.u$. Also since hx, hy, hz, &c. are the quantities by which the variables x, y, z, &c. are increased, the function becomes by Art. 107,

$$u + \partial_{(x)}u.hx + \partial_{(y)}u.hy + \partial_{(z)}u.hz + \text{\&c.}$$

Hence $(1+h)^m.u = u + (x.\partial_{(x)}u + y.\partial_{(y)}u + z.\partial_{(z)}u + \text{\&c.}).h + \dots$ therefore equating the coefficients of h on each side (*Wood's* Alg. Art. 346),

$$mu = x.\partial_{(x)}u + y.\partial_{(y)}u + z.\partial_{(z)}u + \text{\&c.},$$

or, ***the function multiplied by the number indicating its dimensions is equal to the sum of the quantities formed by multiplying each partial differential of the function by its corresponding variable.*** (*EULER'S THEOREM.*)*

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* This Theorem may be proved more directly in the following manner:—

Let u be a homogeneous function of x, y, z, &c. of m dimensions, and assume each of the quantities x, y, z, &c. to be a function of t, or $x = pt$, $y = qt$, $z = rt$, &c. Then if these values of x, y, z, &c. be substituted in the function u, the result will evidently be of the form $t^m.f(p, q, r, \text{\&c.})$. Now if x, y, z, &c. be, or be supposed, independent of each other,

$$\begin{array}{ll} \partial_t\{t^m.f(p, q\ r, \text{\&c.})\} \text{ will be} & = m t^{m-1}.f(p, q, r, \text{\&c.}), \\ \partial_t x \quad \dots\dots\dots & = p, \\ \partial_t y \quad \dots\dots\dots & = q, \\ \partial_t z \quad \dots\dots\dots & = r, \\ \text{\&c.} & \text{\&c.} \end{array}$$

But whatever be the relations between x, y, z, &c.,

$$\partial_t u = \partial_{(x)}u.\partial_t x + \partial_{(y)}u.\partial_t y + \partial_{(z)}u.\partial_t z + \text{\&c. (Art. 72)};$$

$$\therefore\ m t^{m-1}.f(p, q, r, \text{\&c.}) = p.\partial_{(x)}u + q.\partial_{(y)}u + r.\partial_{(z)}u + \text{\&c.}$$

$$m t^m.f(p, q, r, \text{\&c.}) = pt.\partial_{(x)}u + qt.\partial_{(y)}u + rt.\partial_{(z)}u + \text{\&c.}$$

$$\text{or} \qquad mu = x.\partial_{(x)}u + y.\partial_{(y)}u + z.\partial_{(z)}u + \text{\&c.}$$

Ex. Required to verify the preceding Theorem, when $u = \frac{xyz}{x + y + z}$, an homogeneous function of two dimensions.

$$u = \frac{xyz}{x + y + z},$$

$$\partial_{(x)} u = \frac{(x + y + z) yz - zyz}{(x + y + z)^2}$$

$$= \frac{(y + z) yz}{(x + y + z)^2};$$

$$\text{Similarly } \partial_{(y)} u = \frac{(x + z) xz}{(x + y + z)^2},$$

$$\text{and } \partial_{(z)} u = \frac{(x + y) xy}{(x + y + z)^2};$$

Therefore, multiplying and adding,

$$x.\partial_{(x)} u + y.\partial_{(y)} u + z.\partial_{(z)} u = \frac{2(x + y + z) xyz}{(x + y + z)^2}$$

$$= 2 \frac{xyz}{x + y + z}$$

$$= 2u.$$

109. *To expand a function of two variables in positive ascending and integral powers of the variables, when the function is capable of such expansion.*

Let u represent the function of x and y, and assume the expansion to be

$$\begin{aligned} u = a_0 + a_1 x + a_2 x^2 &+ \text{\&c.} \\ + b_1 y + c_2 xy &+ \text{\&c.} \\ + b_2 y^2 &+ \text{\&c.} \\ &+ \text{\&c.}; \end{aligned}$$

then it is evident that $\partial^n_{(x)}u$ will be equal to $\underline{|n}.a_n$ with other terms containing positive integral powers of x or y, or of both, as factors; and consequently, substituting 0 for x and y,

$$(\partial^n_{(x)}u)_{\substack{x=0\\y=0}} = \underline{|n}.a_n;$$

$$\therefore\ a_n = (\partial^n_{(x)}u)_{\substack{x=0\\y=0}}.\frac{1}{\underline{|n}}.$$

$$\text{Similarly } b_n = (\partial^n_{(y)}u)_{\substack{x=0\\y=0}}.\frac{1}{\underline{|n}}.$$

And, if C_{m+n} be the coefficient of $x^m.y^n$, in the same manner

$$(\partial^m_{(x)}\partial^n_{(y)}u)_{\substack{x=0\\y=0}} = \underline{|m}.\underline{|n}.C_{m+n};$$

$$\text{or, } C_{m+n} = (\partial^m_{(x)}\partial^n_{(y)}u)_{\substack{x=0\\y=0}}.\frac{1}{\underline{|m}}.\frac{1}{\underline{|n}}.$$

Hence, substituting these values of the coefficients in the assumed series,

$$\begin{aligned} u = u_{\substack{x=0\\y=0}} &+ (\partial_{(x)}u)_{\substack{x=0\\y=0}}.\frac{x}{\underline{|1}} + (\partial^2_{(x)}u)_{\substack{x=0\\y=0}}.\frac{x^2}{\underline{|2}} + \&c. \\ &+ (\partial_{(y)}u)_{\substack{x=0\\y=0}}.\frac{y}{\underline{|1}} + (\partial_{(x)}\partial_{(y)}u)_{\substack{x=0\\y=0}}.\frac{x.y}{\underline{|1}.\underline{|1}} + \&c. \\ &+ (\partial^2_{(y)}u)_{\substack{x=0\\y=0}}.\frac{y^2}{\underline{|2}} + \&c. \\ &+ \&c. \end{aligned}$$

This method of obtaining the expansion may evidently be extended to functions of a greater number of variables.

The preceding expansion may be deduced from that proved in Art. 107, in the same manner as Maclaurin's Theorem was deduced from Taylor's in Art. 105, i. e. by substituting first 0 for x and y, and then x and y for h and k.

110. *If u be a function of t and v, given by the equation* $u = v + t.\phi u$, *to expand any function of u, as fu, in a series of ascending powers of t.* [LAGRANGE'S THEOREM.]

Let fu be represented by y, and ϕu by z;

then since $\quad u = v + tz$,

$$\partial_{(v)}u = 1 + t.\partial_{(u)}z.\partial_{(v)}u, \qquad \therefore\ \partial_{(v)}u\,(1 - t.\partial_{(u)}z) = 1,$$

$$\partial_{(t)}u = z + t.\partial_{(u)}z.\partial_{(t)}u, \qquad \therefore\ \partial_{(t)}u\,(1 - t.\partial_{(u)}z) = z.$$

Hence, $\partial_{(t)}u = z.\partial_{(v)}u$,

$\therefore\ \partial_{(u)}z.\partial_{(t)}u = z.\partial_{(u)}z.\partial_{(v)}u$,

or, $\partial_{(t)}z = z.\partial_{(v)}z$.

Similarly, $\partial_{(t)}y = z.\partial_{(v)}y$.

Now, $\partial_{(t)}(z^n.\partial_{(v)}y) = nz^{n-1}.\partial_{(t)}z.\partial_{(v)}y + z^n.\partial_{(t)}\partial_{(v)}y$,

$$\begin{aligned}\therefore \text{ by substitution, } &= nz^{n-1}.z.\partial_{(v)}z.\partial_{(v)}y + z^n.\partial_{(v)}(z.\partial_{(v)}y)\\ &= nz^n.\partial_{(v)}z.\partial_{(v)}y + z^n(\partial_{(v)}z.\partial_{(v)}y + z.\partial^2_{(v)}y)\\ &= \overline{n+1}.z^n.\partial_{(v)}z.\partial_{(v)}y + z^{n+1}.\partial^2_{(v)}y\\ &= \partial_{(v)}(z^{n+1}.\partial_{(v)}y).\end{aligned}$$

Also, $\partial_{(t)}y = z.\partial_{(v)}y$,

$\therefore\ \partial^2_{(t)}y = \partial_{(t)}(z.\partial_{(v)}y) = \partial_{(v)}(z^2.\partial_{(v)}y)$.

$\partial^3_{(t)}y = \partial_{(t)}\partial_{(v)}(z^2.\partial_{(v)}y) = \partial_{(v)}\partial_{(v)}(z^3.\partial_{(v)}y) = \partial^2_{(v)}(z^3.\partial_{(v)}y)$.

And it is evident that the succeeding differentials of y with regard to t will follow the same law;

$$\therefore\ \partial^n_{(t)}y = \partial^{n-1}_{(v)}(z^n.\partial_{(v)}y),$$

$$\text{or, } \partial^n_{(t)}fu = \partial^{n-1}_{(v)}(\overline{\phi u}|^n.\partial_{(v)}fu).$$

But this latter quantity being obtained by treating t as a constant will be the same whether 0 is substituted for t, or v for u, before or after the differentiation; hence

$$(\partial^{n}_{t} fu)_{t=0} = \partial^{n-1}_{(v)} (\overline{\phi v}|^{n} . \partial_{(v)} fv).$$

And by Maclaurin's Theorem, fu being treated as a function of t,

$$fu = (fu)_{t=0} + (\partial_{(t)} fu)_{t=0} . \frac{t}{1} + \&c.$$

$$\therefore\ fu = fv + \phi v . \partial_{(v)} fv . \frac{t}{1} + \partial_{(v)} (\overline{\phi v}|^{2} . \partial_{(v)} fv) . \frac{t^{2}}{\underline{|2}}$$

$$+ \partial^{2}_{(v)} (\overline{\phi v}|^{3} . \partial_{(v)} fv) . \frac{t^{3}}{\underline{|3}} + \&c.$$

Cor. If $fu = u$, $fv = v$, therefore $\partial_{(v)} fv = 1$; and

$$u = v + \phi v . \frac{t}{1} + \partial_{(v)} \overline{\phi v}|^{2} . \frac{t^{2}}{\underline{|2}} + \partial^{2}_{(v)} \overline{\phi v}|^{3} . \frac{t^{3}}{\underline{|3}} + \ldots\ldots\ldots$$

111. *If $u = F(v + t.\phi u)$ to expand fu in a series of ascending powers of t.* [*Laplace's Theorem.*]

Since $\quad u = F(v + t.\phi u)$

$$\partial_{(v)} u = \partial_{(u)} F(v + t.\phi u) . \partial_{(v)} (v + t.\phi u)$$

$$= \partial_{(u)} F(v + t.\phi u) . (1 + t . \partial_{(u)} \phi u . \partial_{(v)} u).$$

Similarly, $\partial_{(t)} u = \partial_{(u)} F(v + t.\phi u) . (\phi u + t . \partial_{(u)} \phi u . \partial_{(t)} u)$,

$$\therefore\ \partial_{(t)} u = \phi u . \partial_{(v)} u.$$

And it may be shewn, as in Art. 110, that,

$$\partial^{n}_{(t)} fu = \partial^{n-1}_{(v)} (\overline{\phi u}|^{n} . \partial_{(v)} fu),$$

and $\quad (\partial^{n}_{(t)} fu)_{t=0} = \partial^{n-1}_{(v)} (\overline{\phi V}|^{n} . \partial_{(v)} fV).$

V representing Fv, the value of $u_{t=0}$;

∴ by Maclaurin's Theorem,

$$fu = fV + (\phi V . \partial_{(v)} fV) . \frac{t}{1} + \partial_{(v)} (\overline{\phi V}|^2 . \partial_{(v)} fV) . \frac{t^2}{\underline{|2}}$$

$$+ \partial^2_{(v)} (\overline{\phi V}|^3 . \partial_{(v)} fV) . \frac{t^3}{\underline{|3}} + \dots\dots$$

112. The Theorems proved in Arts. 107—111 are subject to the same limitations as the Theorems of Taylor and Maclaurin from which they are deduced. They serve to determine the expansions of functions in *all cases* where it is known *independently* that the functions are capable of the required expansion; or where the quantities, according to the powers of which the functions are to be expanded, are indefinitely diminished.

EXAMPLES TO LAGRANGE'S AND LAPLACE'S THEOREMS.

Ex. 1. Required to expand x in terms of p and q; x, p, and q being connected by the equation $x^3 - px + q = 0$; i. e. to approximate to a root of a cubic equation.

Here $x = \frac{q}{p} + \frac{1}{p} . x^3$; and, by Lagrange's Theorem,

if $u = v + t . \phi u$,

$$u = v + \phi v . \frac{t}{1} + \partial_{(v)} \overline{\phi v}|^2 . \frac{t^2}{\underline{|2}} + \partial^2_{(v)} \overline{\phi v}|^3 . \frac{t^3}{\underline{|3}} + \dots\dots$$

In this Ex., x corresponds to u in the Theorem,

$\frac{q}{p}$ .... v ....

$\frac{1}{p}$ .... t ....

and x^3 .... ϕu or ϕx.........

∴ by substitution,

$$x = \frac{q}{p} + \frac{q^3}{p^3}.\frac{1}{p} + \partial_{(v)} v^6 . \frac{1}{1.2p^2} + \partial^2_{(v)} v^9 . \frac{1}{1.2.3p^3} + \ldots\ldots$$

$$= \frac{q}{p} + \frac{q^3}{p^4} + \frac{1.6}{1.2}.\frac{q^5}{p^7} + \frac{1.8.9}{1.2.3}.\frac{q^7}{p^{10}} + \ldots\ldots$$

$$= \frac{q}{p}\left\{1 + \frac{q^2}{p^3} + 3\frac{q^4}{p^6} + 12\frac{q^6}{p^9} + \ldots\ldots\right\},$$

which gives an approximation to the value of one of the three roots of the equation $x^3 - px + q = 0$. Lagrange has shewn that this is the least root; vide *Equations Numeriques*, *Note XI.*

Ex. 2. $u = nt + e . \sin u$; required u in a series of ascending powers of e.

In this Example u corresponds to u in Lagrange's Theorem.

nt .... v ....

e .... t ....

$\sin u$..............ϕu.........................

$$\therefore\ u = nt + \sin v . \frac{e}{1} + (\partial_{(v)} \overline{\sin v}|^2) . \frac{e^2}{1.2} + (\partial^2_{(v)} \overline{\sin v}|^3) . \frac{e^3}{1.2.3} + \&c.$$

$$= nt + \sin nt . \frac{e}{1} + 2 . \sin nt . \cos nt . \frac{e^2}{1.2} + 3(\sin nt)^2 . \cos nt . \frac{e^3}{1.2.3} + \ldots$$

Ex. 3. $x^2 - 2x + 1 = 0$; required $\log_{10} x$.

Here $x = \frac{1}{2} + \frac{1}{2}x^2$; and comparing this with the general expression expanded in Art. 110, Cor.

since x corresponds to u,

$\frac{1}{2}$ .... v,

$\frac{1}{2}$ .... t,

x^2 .... ϕu or ϕx,

$\log_{10} x$................ fu or fx;

∴ $\log_{10} v$................ fv,

and v^2.............. ϕv.

But $$fu = fv + \phi v . \partial_{(v)} fv . \frac{t}{1} + \partial_{(v)} (\overline{\phi v}\rceil^{2} . \partial_{(v)} fv) . \frac{t^2}{1.2} + \ldots\ldots$$

$$\therefore \ \log_{10} x = \log_{10} v + v^2 . \frac{1}{\log_\epsilon 10} . \frac{1}{v} . \frac{1}{2} + \partial_{(v)} \left(\frac{1}{\log_\epsilon 10} . v^3 \right) . \frac{1}{2^3} + \ldots\ldots$$

$$= \log_{10} v + \frac{1}{\log_\epsilon 10} \left\{ v . \frac{1}{2} + \frac{3v^2}{1.2} . \frac{1}{2^2} + \frac{4.5 . v^3}{1.2.3} . \frac{1}{2^3} + \ldots\ldots \right\}$$

$$= \log_{10} \frac{1}{2} + \frac{1}{\log_\epsilon 10} \left\{ \frac{1}{2^2} + \frac{3}{1.2.2^4} + \frac{4.5}{1.2.3.2^6} + \ldots\ldots \right\}.$$

Ex. 4. $y = \log_\epsilon (x + x \sin y)$; required ϵ^y.

By Laplace's Theorem, if $u = F(v + t . \phi u)$,

$$fu = fV + \{ \phi V . \partial_{(v)} fV \} \frac{t}{1} + \partial_{(v)} \{ \overline{\phi V}\rceil^{2} . \partial_{(v)} fV \} \frac{t^2}{\lfloor\underline{2}} + \ldots\ldots$$

where V represents Fv the value of $u_{t=0}$.

In the present case from $u = \log_\epsilon (v + t . \sin u)$ it is required to find ϵ^u.

Now $V = \log_\epsilon v$; $\quad \therefore \ fV = e^{\log_\epsilon v} = v$.

$$\partial_{(v)} fV = 1.$$

$$\phi V = \sin V = \sin \log_\epsilon v;$$

$$\therefore \ \phi V . \partial_{(v)} fV = \sin \log_\epsilon v.$$

$$\partial_{(v)} \{ \overline{\phi V}\rceil^{2} . \partial_{(v)} fV \} = \frac{2}{v} \sin \log_\epsilon v . \cos \log_\epsilon v = \frac{\sin \log_\epsilon v^2}{v}.$$

$$\partial^2_{(v)} \{ \overline{\phi V}\rceil^{3} . \partial_{(v)} fV \} = \tfrac{1}{4} \partial^2_{(v)} \{ 3 \sin \log_\epsilon v - \sin \log_\epsilon v^3 \}$$

$$= -\frac{3}{4} \left\{ \frac{\cos \log_\epsilon v + \sin \log_\epsilon v}{v^2} - \frac{\cos \log_\epsilon v^3 + 3 \sin \log_\epsilon v^3}{v^2} \right\};$$

$$\therefore\ \epsilon^{u} = v + \sin\log_{\epsilon} v.\frac{t}{1} + \frac{\sin\log_{\epsilon} v^{2}}{v}.\frac{t^{2}}{\underline{\lvert 2}}$$

$$-\frac{3}{4}\left\{\frac{\cos\log_{\epsilon} v + \sin\log_{\epsilon} v}{v^{2}} - \frac{\cos\log_{\epsilon} v^{3} + 3\sin\log_{\epsilon} v^{3}}{v^{2}}\right\}\frac{t^{3}}{\underline{\lvert 3}}$$

$$+ \&c.$$

or, if the symbols in the question be restored,

$$\epsilon^{y} = z + \sin\log_{\epsilon} z.\frac{x}{1} + \frac{\sin\log_{\epsilon} z^{2}}{z}.\frac{x^{2}}{\underline{\lvert 2}}$$

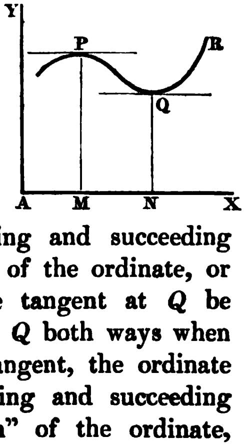
$$-\frac{3}{4}\left\{\frac{\cos\log_{\epsilon} z + \sin\log_{\epsilon} z}{z^{2}} - \frac{\cos\log_{\epsilon} z^{3} + 3\sin\log_{\epsilon} z^{3}}{z^{2}}\right\}\frac{x^{3}}{\underline{\lvert 3}}$$

$$+ \&c.$$

SECTION X.

THE DETERMINATION OF THE MAXIMA AND MINIMA OF FUNCTIONS.

113. Def. "Maxima" or Minima" are those values of a function, which, when the variables are continuously increased or diminished, are respectively greater or less than the immediately preceding and succeeding values.



Thus, if in any continuous plane curve PQR referred to rectangular coordinates AX, AY, the tangent at a point P in the curve be parallel to AX, and the arcs, measured from P both ways, when diminished indefinitely, lie below such tangent, the ordinate MP is greater than the immediately preceding and succeeding ordinates, and therefore is a "maximum" of the ordinate, or a "maximum ordinate."—Again, if the tangent at Q be parallel to AX, and the arcs measured at Q both ways when diminished indefinitely, lie above such tangent, the ordinate NQ is less than the immediately preceding and succeeding ordinates, and therefore is a "minimum" of the ordinate, or a "minimum ordinate".

114. *To determine the "maxima" or "minima" of a function of one variable, if the function can be made to have any such values.*

Let fx represent the function; then $f\overline{x+h}$ and $f\overline{x-h}$, when h is sufficiently diminished, will represent the values of the function immediately preceding and succeeding fx for any particular value of x. Therefore, by the definition, for the values of x which render fx a "maximum" or a "minimum," fx is greater than, or less than, both $f\overline{x+h}$ and $f\overline{x-h}$; or $f\overline{x+h}-fx$ and $f\overline{x-h}-fx$ have the same sign however small h be taken. Hence, by Taylor's Theorem,

$$\partial_x fx.\frac{h}{1}+\partial_x^2 fx.\frac{h^2}{\underline{|2}}+\partial_x^3 fx.\frac{h^2}{\underline{|3}}+\ldots\ldots\ldots\ldots+\partial_x^n fx.\frac{h^n}{\underline{|n}}+\&c.$$

$$\text{and } -\partial_x fx.\frac{h}{1}+\partial_x^2 fx.\frac{h^3}{\underline{|3}}-\partial_x^3 fx.\frac{h^3}{\underline{|3}}+\ldots+(-1)^{n+1}\partial_x^n fx.\frac{h^n}{\underline{|n}}+\&c.$$

must have the same sign for those values of x. But this is impossible unless the first term of each of the series, independently of h, is equal to 0, since by sufficiently diminishing h, (Art. 100, Cor.) the first term can be made greater than the sum of all the succeeding terms; therefore the values of x which give the required values of fx, if there be any, must render $\partial_x fx$ equal to 0, i. e. must be roots of the equation $\partial_x fx = 0$.

Again, since for these values of x,

$$f\overline{x\pm h}-fx=\partial_x^2 fx.\frac{h^2}{\underline{|2}}\pm\partial_x^3 fx.\frac{h^3}{\underline{|3}}+\&c.,$$

and that, by diminishing h, the first term of this series may be made greater than the sum of all the succeeding terms, it is evident that, for such of the roots of $\partial_x fx = 0$ as render $\partial_x^2 fx$ a negative quantity, fx will be greater than the immediately preceding and succeeding values of the function, or fx will be a "maximum." Similarly, for such of the roots as render $\partial_x^2 fx$ positive, fx will be a "minimum."

It is also evident from the above reasoning, that if any root of $\partial_x fx = 0$ render $\partial_x^2 fx$ equal to 0, fx cannot be either a "maximum" or "minimum" for that value, unless it also render $\partial_x^3 fx$ equal to 0; but if it render $\partial_x^3 fx$ equal to 0, then fx will be a "maximum" or "minimum," according as $\partial_x fx$ is negative or positive when such value of x is substituted in it. And it may be stated generally, that those values of x

which render fx a "maximum" or "minimum" must render its successive differentials up to $\partial_x^{2r+1} fx$ inclusive severally equal to 0, and $\partial_x^{2r+2} fx$ a positive or negative quantity; and that such of those values of x as render $\partial_x^{2r+2} fx$ negative give "maxima," and such as render it positive give "minima" of the function fx.

115. If any of the roots of the equation $\partial_x fx = 0$ render the differentials of fx up to $\partial_x^r fx$ inclusive severally equal to 0, and $\partial_x^{r+1} fx$ equal to infinity, it will follow that, for such values of x, $f\overline{x \pm h}$, or $f\overline{x \pm h} - fx$, is incapable of being expanded in positive *integral* powers of h; and that if it can be expanded at all in positive powers of h, the exponent of the first term will be fractional and greater than r. (Vide Art. 102.) For such values of x the expansion of $f\overline{x \pm h} - fx$ must be obtained, when possible, by some other method than by Taylor's Theorem.

Suppose $A(\pm h)^{\alpha} + B(\pm h)^{\beta} +$ &c. to be the expansion so obtained, in which the exponents α, β, &c. are arranged in ascending order of magnitude, α being fractional; then,

If α be of the form $\frac{2m+1}{2n}$, one of the quantities $f\overline{x+h} - fx$ becomes impossible; and therefore although $f\overline{x+h}$ may have two, or any even number of, values, which may be less or greater than fx, fx will not be a "maximum" or "minimum" within the terms of the Definition.

If α be of the form $\frac{2m+1}{2n+1}$, since the first term of the series may be made greater than the sum of all the succeeding terms, and $(\pm h)^{\alpha}$ will have different signs according as $+h$ or $-h$ is taken, fx is neither a "maximum" nor a "minimum."

If α be of the form $\frac{2m}{2n+1}$, since $(\pm h)^{\alpha}$ will have the same sign, whether $+h$ or $-h$ be taken, fx will be a maximum or minimum, according as A is negative or positive, unless any of the succeeding exponents should be of the form $\frac{2m+1}{2n}$, in which case, as before, although fx may have a greatest or least value, it will not be a "maximum" or "minimum."

116. It appears from the preceding Arts. that of all the particular values of x which render $f\overline{x \pm h} - fx$ capable of expansion in ascending positive powers of h, where the first exponent is not less than 1, such of them as make fx a "maximum" or "minimum" must be roots of the equation $\partial_x fx = 0$.

There may, however, be other values of x which render $f\overline{x \pm h} - fx$ capable of expansion in positive ascending powers of h, where the first exponent is less than 1. In such cases (vide Art. 100.) the first differential of the function, or $\partial_x fx$, will be infinite, and therefore all such values must be roots of the equation $\frac{1}{\partial_x fx} = 0$. Hence all the other values of x, if any, which render fx a "maximum" or "minimum" must be roots of this latter equation; and if these roots be successively substituted in the expression $f\overline{x \pm h} - fx$, and the result expanded by any method into a series of the form $A(\pm h)^a + \&c.$, it may be ascertained, as in the last Art., which, if any, of these values give "maxima," and which, if any, give "minima" of the function fx.

117. It is sometimes easy to determine *à priori* that a proposed function does not admit of any "maxima" or "minima."

Thus, if the function be $a + bx^3$, it is evident that, as x is continuously increased, the function also continuously increases, and cannot admit of a "maximum" or "minimum." Again, if $y^2 = \frac{x^3}{2a - x}$, it is evident that, as x is continuously increased from 0 up to $2a$, (beyond which there is no value of x that will give a possible value for y) y^2, and therefore y, continuously increases; consequently, y cannot have a "maximum" or "minimum" value. In such cases it is manifestly useless to apply the preceding Theory, although the same result would be obtained from it.

118. The process of determining "maxima" and "minima" may often be considerably abridged by particular artifices.

Thus, if fx be a "maximum," $\frac{1}{fx}$ will be a "minimum" for the same values of x, and *vice versa;* so that the required values of x may be found among the roots of either $\partial_x fx = 0$, or $\frac{1}{\partial_x fx} = 0$, as is most convenient. Again, it is obvious that $\log fx$, $c.fx$, $(fx)^n$, will severally be "maxima" and "minima" for the same values of x which render fx a "maximum" or "minimum"; and it will frequently be found more convenient to "differentiate" one of the former quantities than the proposed function fx.

Again, if $\partial_x fx = p.q$, p and q being both functions of x, since $\partial_x^2 fx = p.\partial_x q + q.\partial_x p$, the substitution of the roots of $p = 0$, will reduce it to $\partial_x^2 fx = q.\partial_x p$; and the substitution of the roots of $q = 0$, will reduce it to $\partial_x^2 fx = p.\partial_x q$.

Similarly, if $\partial_x fx = \frac{p}{q}$, since $\partial_x^2 fx = \frac{1}{q^2}\{q.\partial_x p - p.\partial_x q\}$, the substitution of the roots of $p = 0$ will reduce it to $\partial_x^2 fx = \frac{\partial_x p}{q}$.

The application of these results will evidently abridge the usual process for finding "maxima" and "minima."

Thus, if $\partial_x fx = \frac{ax(4a - 3x)}{\sqrt{4a^2x^2 - 2ax^3}}$, which may be put under the form $\partial_x fx = (4a - 3x).\frac{a}{\sqrt{4a^2 - 2ax}}$, and it be required to find the value of $\partial_x^2 fx$ when $4a - 3x = 0$, or $x = \frac{4a}{3}$; by the first of the above rules

$$(\partial_x^2 fx)_{x=\frac{4a}{3}} = \left\{\frac{a}{\sqrt{4a^2 - 2ax}} . \partial_x (4a - 3x)\right\}_{x=\frac{4a}{3}}$$

$$= -\frac{3a}{\sqrt{\frac{4a^2}{3}}} = -\frac{3\sqrt{3}}{2}.$$

119. It is obvious from the Definition of "maxima" and "minima" (Art. 109.), that a function may admit of several "maxima" and several "minima," that some of the "minima" may be greater than some of the "maxima," and that a "maximum" is not necessarily the greatest, or a "minimum" the least, value of a function.

In the application of the preceding Theory, however, to Physical Problems, a "maximum" of a function will in general coincide with the greatest, and a "minimum" with the least, value of the function, because the function is such, that it cannot admit of several "maxima," or several "minima," and its value cannot be increased or diminished beyond certain limits.

120. If y be an "*implicit* function" of x given by the equation $\phi(x, y) = 0$, its "maxima" and "minima," if it have any such values, may evidently be determined according to the preceding Theory, by first solving the equation with respect to y, and putting it under the form $y = fx$. And since the *values* of $\partial_x y$, $\partial_x^2 y$, &c. will be the same whether they are obtained from $y = fx$, or from $\phi(x, y) = 0$ according to the method given in Art. 43 or Art. 74, the same Theory will be equally applicable in both cases for determining the "maxima" and "minima" of y.

If the successive differentials of y are found, considering it as an *implicit* function of x, and $\phi(x, y)$ be represented by u, since

$$\partial_x y = -\frac{\partial_{(x)} u}{\partial_{(y)} u},$$

the values of x which render y a "maximum" or "minimum" will be found among those which are obtained by elimination from the equations

$$u = 0, \qquad \partial_{(x)} u = 0,$$

$$\text{or from } u = 0, \qquad \frac{1}{\partial_{(y)} u} = 0.$$

In the first case, for any such values of x,

$$\partial_x^2 y = -\frac{\partial_{(x)}^2 u}{\partial_{(y)} u}. \quad \text{(Vide Art. 74.)}$$

EXAMPLES.

Ex. 1. Required the "maxima" and "minima" of

$$x^5 - 5x^4 + 5x^3 + 1.$$

Let $x^5 - 5x^4 + 5x^3 + 1$ be represented by fx.

$$\text{Then } \partial_x fx = 5x^4 - 20x^3 + 15x^2 = 0;$$

the roots of which equation are 0, 1, 3.

$$\partial_x^2 fx = 20x^3 - 60x^2 + 30x,$$

$$\text{and } \partial_x^3 fx = 60x^2 - 120x + 30.$$

$$\left.\begin{array}{rl} & (\partial_x^2 fx)_{x=0} = 0 \\ \text{and} & (\partial_x^3 fx)_{x=0} = 30 \end{array}\right\} \therefore (fx)_{x=0} \text{ is neither a "maximum" nor a "minimum".}$$

$$\left.\begin{array}{rl} & (\partial_x^2 fx)_{x=1} = -10 \\ \text{and} & (\partial_x^3 fx)_{x=1} = -30 \end{array}\right\} \therefore (fx)_{x=1} = 2 \text{ is a "maximum".}$$

$$\left.\begin{array}{rl} & (\partial_x^2 fx)_{x=3} = 90 \\ \text{and} & (\partial_x^3 fx)_{x=3} = 210 \end{array}\right\} \therefore (fx)_{x=3} = -26 \text{ is a "minimum".}$$

Ex. 2. Required the "maxima" and "minima" of

$$\frac{a+x}{a-x}\cdot\frac{b+x}{b-x}.$$

Since $\log_\epsilon \phi x$ is a "maximum" or "minimum" for the same values of x for which ϕx is, the values of x which render the proposed function a "maximum" or "minimum" will be the same as those which render

$$\log_\epsilon \overline{a+x} - \log_\epsilon.\overline{a-x} + \log_\epsilon \overline{b+x} - \log_\epsilon \overline{b-x}$$

a "maximum" or "minimum;" therefore, representing this latter function by fx.

$$\partial_x fx = \frac{1}{a+x} + \frac{1}{a-x} + \frac{1}{b+x} + \frac{1}{b-x} = 0$$

$$= \frac{2a}{a^2-x^2} + \frac{2b}{b^2-x^2} = 0,$$

the roots of which equation are $\sqrt{ab}$ and $-\sqrt{ab}$.

$$\partial_x^2 fx = \frac{4ax}{(a^2-x^2)^2} + \frac{4bx}{(b^2-x^2)^2},$$

$$\therefore\quad (\partial_x^2 fx)_{x=\sqrt{ab}} = \text{a positive quantity,}$$

$$\text{and}\quad (\partial_x^2 fx)_{x=-\sqrt{ab}} = \text{a negative ....}$$

To find $\partial_x^3 fx$, by Art. 29, $\partial_x \dfrac{4ax}{(a^2-x^2)^2} = \dfrac{a^2+3x^2}{x(a^2-x^2)^3}$,

$\therefore\quad \partial_x^3 fx = \dfrac{a^2+3x^2}{x(a^2-x^2)^3} + \dfrac{b^2+3x^2}{x(b^2-x^2)^3}$, which remains finite whether $\sqrt{ab}$ or $-\sqrt{ab}$ be substituted in it for x.

Hence, $\left(\dfrac{a+x}{a-x}\cdot\dfrac{b+x}{b-x}\right)_{x=\sqrt{ab}} = -\left(\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2$ is a "minimum",

and $\left(\dfrac{a+x}{a-x}\cdot\dfrac{b+x}{b-x}\right)_{x=-\sqrt{ab}} = -\left(\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)^2$ is a "maximum".

Ex. 3. Required the "maxima" and "mini
x and y being connected by the equation

$$x^3 + y^3 - 3axy = 0.$$

By Art. 74. Ex., $\partial_x y = -\dfrac{x^2 - ay}{y^2 - ax}$;

$$\therefore\ x^2 - ay = 0, \text{ or } y = \frac{x^2}{a}.$$

Substituting this value of y in the original equation,

$$x^3 + \frac{x^6}{a^3} - 3x^3 = 0,$$

$$\text{or } x^6 - 2a^3x^3 = 0,$$

the roots of which equation are 0, and $a\sqrt[3]{2}$.

$$\begin{aligned}
\text{Again, (Art. 74.), } \partial_x^2 y &= -\frac{2a^3xy}{(y^2 - ax)^3};\\
&= -\frac{2a^2x^3}{\left(\dfrac{x^4}{a^2} - ax\right)^3};\quad \because\ y = \frac{x^2}{a},\\
&= -\frac{2a^8}{(x^3 - a^3)^3};
\end{aligned}$$

$$\therefore\ (\partial_x^2 y)_{x=0} = \frac{2}{a}, \text{ a positive quantity,}$$

$$\text{and } (\partial_x^2 y)_{x=a\sqrt[3]{2}} = -\frac{2}{a}, \text{ a negative } \ldots\ldots\ldots;$$

$$\therefore\ y_{x=0} = 0 \text{ is a minimum,}$$

$$\text{and } y_{x=a\sqrt[3]{2}} = 2^{\frac{2}{3}}.a \text{ is a maximum.}$$

Ex. 4. Required the "maxima" and "minima" of

$$b + (x - a)^{\frac{2}{3}}.$$

Let fx represent the proposed function,

$$\text{then } \partial_x fx = \tfrac{3}{2}(x-a)^{\frac{1}{2}} = 0,$$

which is satisfied by $x = a$.

$$\partial_x^2 fx = \tfrac{3}{4}(x-a)^{-\frac{1}{2}},$$

$$\text{and } (\partial_x^2 fx)_{x=a} = \infty .$$

Also $(\partial_x^3 fx)_{x=a}$, and all the successive differentials severally become ∞ when $x = a$.

$$\left.\begin{array}{l}\text{Now } f\overline{a+h} - fa = h^{\frac{3}{2}}, \\ \text{and } f\overline{a-h} - fa = (-h)^{\frac{3}{2}}\end{array}\right\} \text{ the latter of which results is impossible;}$$

Hence, (as stated generally in Art. 115.) although $(fx)_{x=a}$ is evidently the *least* possible value of fx, it is not a "*minimum.*"

It is evident that the proposed function can have no "maximum," because, as x is continuously increased, fx continuously increases *ad infinitum.*

Ex. 5. Required the "maxima" and "minima" of

$$b + c(x-a)^{\frac{2}{3}}.$$

Let fx represent the proposed function,

$$\text{then } \partial_x fx = \tfrac{2}{3}(x-a)^{-\frac{1}{3}},$$

and if $x = a$, $\partial_x fx$, $\partial_x^2 fx$, &c. severally become ∞.

$$\text{Now } f\overline{a+h} - fa = ch^{\frac{2}{3}},$$

$$\text{and } f\overline{a-h} - fa = ch^{\frac{2}{3}};$$

both of which results are positive, if c be positive, and negative, if c be negative; therefore

$(fx)_{x=a} = b$ is a "minimum," if c be positive,

and $(fx)_{x=a} = b$ is a "maximum," if c be negative.

121. When the ***greatest*** or ***least*** value of a Geometrical or Physical function (which in general coincides with a "maximum" or "minimum" of the function), is required, the method of applying the preceding Theory is, first to assume one or more variables at discretion to represent certain quantities, which are not given by the question, and upon which the magnitude of the function depends, and then to express the function, either explicit or implicit, in terms of these variables and given quantities. Having thus obtained a mathematical expression for the function, its "maximum" or "minimum" may be found as in the preceding Examples.

Ex. 1. Required the greatest and least ordinates of a circle.

Let x and y be the co-ordinates to any point of the circle,

a and b .... the centre ....,

r .... radius;

then $(x-a)^2+(y-b)^2=r^2$ (Hymers' Conic Sections, Art. 46.);

$$\therefore\ x-a+(y-b).\partial_x y=0,$$

$$\text{or}\ \partial_x y=-\frac{x-a}{y-b}=0,$$

$$\therefore\ x-a=0,\ \text{or}\ x=a.$$

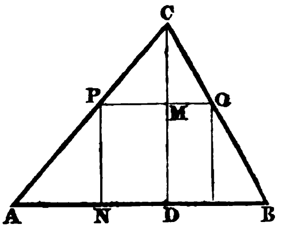
$$\text{Again,}\ 1+(\partial_x y)^2+(y-b).\partial_x^2 y=0;$$

$$\therefore\ (\partial_x^2 y)_{x=a}=-\frac{1}{\pm r};$$

$$\therefore\ y_{x=a}=b+r\ \text{is a "maximum,"}$$

$$\text{and}\ y_{x=a}=b-r\ \text{is a "minimum."}$$

Ex. 2. Required to cut out from a given plane triangle the greatest rectangular parallelogram, having one side coincident with the base of the triangle.



Let ABC be the given triangle; AB, the base, $= c$; NQ the greatest rectangular parallelogram required; CD, the altitude of the triangle, $= h$; $CM = x$.

Then $PQ.PN$ is a maximum.

$$\text{Now } \frac{PQ}{AB} = \frac{CM}{CD};$$

$$\therefore PQ = \frac{cx}{h},$$

$$\text{and } PN = h - x;$$

$$\therefore \frac{cx}{h}.(h-x) \text{ is a ``maximum,''}$$

or (Art. 118.) $hx - x^2 = fx$, a "maximum;"

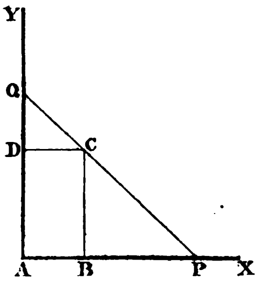
$$\therefore \partial_x fx = h - 2x = 0, \text{ and } \therefore x = \frac{h}{2};$$

$$\text{Also, } \partial_x^2 fx = -2.$$

Hence the greatest rectangular parallelogram required is cut out by bisecting the sides.

Also, its magnitude $= \frac{c}{2}.\frac{h}{2} = \frac{1}{2}.$area of the triangle.

Ex. 3. AX, AY are two straight lines at right angles to each other, C is a given fixed point, required the shortest straight line which can be drawn through C, terminated by AX and AY.



Let PCQ be the required line; AB (the distance of C from AY) $= a$,

BC (the distance of C from AX) $= b$,

$\angle APC = x$.

Then $PC = b.\text{cosec}\, x$,

$$QC = a.\sec x\,;$$

$$\therefore\ PQ = a.\sec x + b.\text{cosec}\, x = fx, \text{ a "minimum;"}$$

$$\therefore\ \partial_x fx = a.\tan x.\sec x - b.\cot x.\text{cosec}\, x$$

$$= \frac{a\,(\sin x)^3 - b\,(\cos x)^3}{(\sin x)^2.\,(\cos x)^2} = 0\,;$$

$$\therefore\ \frac{\sin x}{\cos x} \text{ or } \tan x = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}},$$

$$\text{and } \sin x = \frac{\tan x}{\sec x} = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}\sqrt{1 + \dfrac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}}} = \frac{b^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}\,;$$

$$\therefore\ PQ = a.\sqrt{1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}} + b^{\frac{2}{3}}.\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}$$

$$= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

A "*maximum*" value of PQ is clearly inadmissible *à priori*.

This is the same problem as the following:—To determine the longest straight pole which can be put up a given common chimney in a plane at right angles to the back of the chimney.

Ex. 4. Required the cylindrical mug of given content which shall have the least internal surface.

Let x be the radius of the base,

C the given content,

$$\text{then the height of the mug} = \frac{\text{content}}{\text{base}}$$

$$= \frac{C}{\pi x^2}\,;$$

$$\therefore \text{ the internal surface} = \pi x^2 + 2\pi x.\frac{C}{\pi x^2}$$

$$= \pi x^2 + \frac{2C}{x} = fx, \text{ a "minimum"};$$

$$\therefore\ 2\pi x - \frac{2C}{x^2} = \partial_x fx = 0;$$

$$\therefore\ x = \sqrt[3]{\frac{C}{\pi}};$$

which gives a "minimum", and not a "maximum", since $\partial^2_x fx = 2\pi + \frac{4C}{x^3}$, a positive quantity, when $x^3 = \frac{C}{\pi}$.

$$\text{The height of the mug} = \frac{C}{\pi.\frac{C^{\frac{2}{3}}}{\pi^{\frac{2}{3}}}} = \sqrt[3]{\frac{C}{\pi}} = \text{radius of base.}$$

Ex. 5. Required the greatest plane area which can be included by four given straight lines taken in a given order.

Let a, b, c, d represent the lengths of the given lines; and when the included area is the greatest, let θ, contained by a and b, and ϕ, contained by c and d, be two opposite angles of the quadrilateral; D the diagonal subtending each of these angles.

Then, since the quadrilateral is equal to the two triangles into which it is divided by D,

$$\text{its area} = \frac{ab}{2}.\sin\theta + \frac{cd}{2}.\sin\phi = f\theta,$$

a function of one variable (θ) only, since θ and ϕ are connected by the equation,

$$a^2 + b^2 - 2ab.\cos\theta = D^2$$

$$= c^2 + d^2 - 2cd.\cos\phi \ldots \text{(i)}.$$

Therefore, differentiating on this supposition,

$$\partial_\theta f\theta = ab.\cos\theta + cd.\cos\phi.\partial_\theta\phi = 0.$$

But from (i) $ab.\sin\theta - cd.\sin\phi.\partial_\theta\phi = 0,$

$$\therefore\ ab\,(\cos\theta.\sin\phi + \sin\theta.\cos\phi) = 0;$$

$$\text{or,}\quad \sin\overline{\theta+\phi} = 0,$$

$$\therefore\ \theta+\phi = 180^\circ\ldots\ldots\text{(ii)}.$$

Hence the required quadrilateral is that which can be inscribed in a circle; and θ and ϕ may be found from (i) and (ii). If necessary, it may easily be shewn that

$$(\partial^2_\theta f\theta)_{\theta+\phi=180^\circ} = -ab - cd, \text{ a negative quantity.}$$

122. *To determine the "maxima" or "minima" of a function of two independent variables.*

Let $u\{=f(x,y)\}$ represent the function; then, by the Definition, for the values of x and y which render u a "maximum" or "minimum", x and y being independent variables, $f(\overline{x+h}, \overline{y+k}) - f(x,y)$ must retain the same sign, however small h and k be taken, and whether they are positive or negative, and have the same or different signs. Therefore, if $k = mh$, by Art. 107.

$$\begin{aligned}
&\partial_{(x)}u.\frac{h}{1} + \partial^2_{(x)}u.\frac{h^2}{\underline{|2}} + \ldots\ldots\ldots \\
&+ \partial_{(y)}u.\frac{mh}{1} + \partial_{(x)}\partial_{(y)}u.\frac{mh^2}{1} + \ldots\ldots\ldots \\
&\qquad + \partial^2_{(y)}u.\frac{m^2h^2}{\underline{|2}} + \ldots\ldots\ldots \\
&\qquad\qquad + \ldots\ldots\ldots
\end{aligned}$$

must retain the same sign, however small h be taken, and whether it be positive or negative. But this is impossible unless the first column of the expansion is equal to 0, or, since m is an arbitrary quantity, unless $\partial_{(x)}u$ and $\partial_{(y)}u$ are separately equal to 0. Hence the values of x and y, which render $f(x,y)$ a "maximum" or "minimum", may be found among the roots of the equations $\partial_{(x)}f(x,y)=0$, $\partial_{(y)}f(x,y)=0$.

Again, for those roots of the equations $\partial_{(x)}f(x,y)=0$, $\partial_{(y)}f(x,y)=0$, which render $f(x,y)$ a "maximum" or "minimum", the second column of terms must retain the same sign, whatever be the value of m; therefore, if A, B, C represent $\partial^2_{(x)}u$, $\partial_{(y)}\partial_{(y)}u$, $\partial^2_{(y)}u$ respectively, $Am^2+2Bm+C$ must be incapable of changing its sign for any value whatever of m; but this expression may be put under the form

$$A\left\{\left(m+\frac{B}{A}\right)^2+\frac{AC-B^2}{A^2}\right\},$$

and, since $\left(m+\frac{B}{A}\right)^2$ is always positive, if A and C have the same sign and AC be $> B^2$, $Am^2+2Bm+C$ will always have the same sign as A, whatever m be, and will not change that sign by any change of m.

Hence the values of x and y, which satisfy the equations $\partial_{(x)}u=0$, $\partial_{(y)}u=0$, and also the condition

$$\partial^2_{(x)}u.\partial^2_{(y)}u > (\partial_{(x)}\partial_{(y)}u)^2,$$

will render u a "maximum" or "minimum", according as they render $\partial^2_{(x)}u$ and $\partial^2_{(y)}u$ (which must necessarily have the same sign) negative or positive.

It is evident from the above reasoning, that, if any of the values of x and y, which are roots of the equations $\partial_{(x)}u=0$, $\partial_{(y)}u=0$, render $\partial^2_{(x)}u$, $\partial_{(x)}\partial_{(y)}u$, and $\partial^2_{(y)}u$ separately equal to 0, u cannot be a "maximum" or "minimum" for those values, unless the differentials of the next higher order are also rendered separately equal to 0, and the next higher than these remain finite. And so on, proceeding, if necessary, to higher orders of differentials, i.e. taking more terms of the expansion.

123. Whatever be the number of the variables x, y, z, &c. of which the proposed quantity u is a function, it is clear, from the reasoning employed in the preceding Art., that its "maxima" or "minima," if it have any, will be found among the roots of the equations $\partial_{(x)}u = 0$, $\partial_{(y)}u = 0$, $\partial_{(z)}u = 0$, &c.:—but the determination of the necessary conditions, and the application of them to particular examples, increases in difficulty with the number of variables.

In the case of a function u of *three* independent variables x, y, z, the conditions of "maxima" and "minima" will be found to be,

(1)......$\partial_{(x)}u = 0$, $\partial_{(y)}u = 0$, $\partial_{(z)}u = 0$.

(2)......$\partial^2_{(x)}u$, $\partial^2_{(y)}u$, $\partial^2_{(z)}u$, having the same sign when the roots of the first three equations are substituted in them.

$$(3)\ldots\ldots\begin{cases} \partial^2_{(x)}u.\partial^2_{(y)}u > (\partial_{(x)}\partial_{(y)}u)^2 \\ \partial^2_{(x)}u.\partial^2_{(z)}u > (\partial_{(x)}\partial_{(z)}u)^2 \\ \partial^2_{(y)}u.\partial^2_{(z)}u > (\partial_{(y)}\partial_{(z)}u)^2, \end{cases}$$

and (4)

$$\partial^2_{(x)}u.\partial^2_{(y)}u-(\partial_{(x)}\partial_{(y)}u)^2\}.\{\partial^2_{(x)}u.\partial^2_{(z)}u-(\partial_{(y)}\partial_{(z)}u)^2\}>\{\partial_{(x)}\partial_{(y)}u.\partial_{(x)}\partial_{(z)}u-\partial_{(y)}\partial_{(z)}u.\partial^2_{(x)}u\}^2.$$

Since the conditions (3) are deduced from the last condition, if this hold, they will necessarily hold also; but as the last cannot hold if any one of (3) fail, it is expedient to test them first.

Ex. 1. Required the "maxima" or "minima" of

$$xy + \frac{a^3}{x} + \frac{a^3}{y}.$$

Let the proposed function be represented by u,

$$\text{then } \partial_{(x)}u = y - \frac{a^3}{x^2} = 0\ldots\ldots\ldots(1).$$

$$\partial_{(y)}u = x - \frac{a^3}{y^2} = 0\ldots\ldots\ldots(2);$$

$$\therefore\ x^2y = xy^2 = a^3,\quad \text{or } x = y = a.$$

$$\text{Also, } \partial^2_{(x)}u = \frac{2a^3}{x^3}; \qquad \therefore \quad (\partial^2_{(x)}u)_{x=a} = 2.$$

$$\partial^2_{(y)}u = \frac{2a^3}{y^3}; \qquad \therefore \quad (\partial^2_{(y)}u)_{y=a} = 2.$$

$$\partial_{(x)}\partial_{(y)}u = 1; \qquad \therefore \quad (\partial_{(x)}\partial_{(y)}u)_{\substack{x=a\\y=a}} = 1.$$

Hence, since $\partial^2_{(x)}u.\partial^2_{(y)}u > (\partial_{(x)}\partial_{(y)}u)^2$ for the values of x and y derived from (1) and (2), and $\partial^2_{(x)}u$, $\partial^2_{(y)}u$ are both positive for the same values,

$$u_{\substack{x=a\\y=a}} = 3a^2, \text{ is a “minimum.”}$$

Ex. 2. Required the “maxima” or “minima” of

$$ax + by + cz,$$

subject to the condition that $x^2 + y^2 + z^2 = r^2$.

Let $ax + by + cz$ be represented by u; then, considering z as a function of x and y,

$$\partial_{(x)}u = a + c.\partial_{(x)}z = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

$$\partial_{(y)}u = b + c.\partial_{(y)}z = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

But, since $x^2 + y^2 + z^2 = r^2$,

$$x + z.\partial_{(x)}z = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

$$\text{and } y + z.\partial_{(y)}z = 0 \ldots\ldots\ldots\ldots\ldots (4);$$

therefore, eliminating $\partial_{(x)}z$ and $\partial_{(y)}z$ between (1), (2), (3) and (4),

$$az - cx = 0,$$

$$\text{and } bz - cy = 0;$$

$$\therefore \; bx - ay = 0,$$

$$\text{and } x^2 + \frac{b^2x^2}{a^2} + \frac{c^2x^2}{a^2} = r^2;$$

$$\text{or}\quad x = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}.r.$$

$$\text{Similarly}\quad y = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}.r\,;$$

$$\text{and}\quad z = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.r.$$

$$\text{Again,}\quad \left.\begin{aligned} \partial^2_{(x)}u &= c.\partial^2_{(x)}z \\ \partial^2_{(y)}u &= c.\partial^2_{(y)}z \\ \partial_{(x)}\partial_{(y)}u &= c.\partial_{(x)}\partial_{(y)}z \end{aligned}\right\}\text{ from (1) and (2).}$$

Also, from (3) and (4),

$$1 + (\partial_{(x)}z)^2 + z.\partial^2_{(x)}z = 0,$$

$$1 + (\partial_{(y)}z)^2 + z.\partial^2_{(y)}z = 0,$$

$$\text{and}\quad \partial_{(x)}z.\partial_{(y)}z + z.\partial_{(x)}\partial_{(y)}z = 0\,;$$

$$\therefore\quad \partial^2_{(x)}u = -\frac{c}{z}\left\{1 + (\partial_{(x)}z)^2\right\}$$

$$= -\frac{c}{z^3}\left\{x^2 + z^2\right\}$$

$$\partial^2_{(y)}u = -\frac{c}{z^3}\left\{y^2 + z^2\right\}$$

$$\partial_{(x)}\partial_{(y)}u = -\frac{c}{z^3}.xy.$$

And, substituting in the last three equations the values of x, y, z, found above,

$$\partial^2_{(x)}u \text{ and } \partial^2_{(y)}u \text{ are both positive;}$$

$$\text{and}\quad \partial^2_{(x)}u.\partial^2_{(y)}u \;:\; (\partial_{(x)}\partial_{(y)}u)^2 = (a^2+c^2).(b^2+c^2) \;:\; a^2b^2,$$

which is evidently a "ratio of greater inequality".

Hence, the values of x, y, z, found above, render the proposed function a "*minimum*", and its value is

$$r.\sqrt{a^2+b^2+c^2}.$$

Ex. 3. Amongst all rectangular parallelopipedoms of given volume, required to find that which has the least surface.

Let x, y, z be the length, breadth and depth respectively: and V the given volume; then

$$\text{the surface} = 2xy + 2xz + 2yz;$$

$\therefore\ xy + xz + yz = u$ a "minimum", subject to the condition

$$xyz = V \ldots\ldots\ldots\ldots\ldots (a),$$

and considering z as a function of x and y,

$$\partial_{(x)}u = y + z + (x+y).\partial_{(x)}z = 0 \ldots\ldots (1)$$

$$\partial_{(y)}u = x + z + (x+y).\partial_{(y)}z = 0 \ldots\ldots (2);$$

Also, from (a), $yz + xy.\partial_{(x)}z = 0 \ldots\ldots (3)$

$$xz + xy.\partial_{(y)}z = 0 \ldots\ldots (4);$$

$\therefore$ eliminating $\partial_{(x)}z$, and $\partial_{(y)}z$ between (1), (2), (3), and (4),

$$x - z = 0, \text{ and } y - z = 0;$$

$$\therefore\ x = y = z = \sqrt[3]{V}.$$

Again, $\partial^2_{(x)}u = 2.\partial_{(x)}z + (x+y).\partial^2_{(x)}z$

$$\partial^2_{(y)}u = 2.\partial_{(y)}z + (x+y).\partial^2_{(y)}z$$

$$\partial_{(x)}\partial_{(y)}u = 1 + \partial_{(x)}z + \partial_{(y)}z + (x+y).\partial_{(x)}\partial_{(y)}z.$$

Also, from (3) and (4),

$$2.\partial_{(x)}z + x.\partial^2_{(x)}z = 0$$

$$2.\partial_{(y)}z + y.\partial^2_{(y)}z = 0$$

$$\partial_{(y)}z + x.\partial_{(x)}\partial_{(y)}z = 0;$$

$$\therefore\ \partial^2_{(x)}u = y.\partial^2_{(x)}z$$

$$= \frac{2yz}{x^2}.$$

$$\partial^2_{(y)}u = \frac{2xz}{y^2};$$

$$\partial_{(x)}\partial_{(y)}u = 1 - \frac{z}{x} - \frac{z}{y} + (x+y).\frac{z}{xy}.$$

And, substituting in the last three equations the values of x, y, z, found above,

$$\partial^2_{(x)}u \text{ and } \partial^2_{(y)}u \text{ are both positive,}$$

and $\partial^2_{(x)}u.\partial^2_{(y)}u : (\partial_{(x)}\partial_{(y)}u)^2 = 4 : 1$, a "ratio of greater inequality".

Hence, $x=y=z=\sqrt[3]{V}$ renders u a "*minimum*"; or the parallelopidedom required must be a cube whose edge $=\sqrt[3]{V}$.

SECTION XI.

CURVES.

I. DETERMINATION OF THE DIRECTION OF CURVATURE, AND OF CERTAIN REMARKABLE POINTS AND PROPERTIES OF PLANE CURVES.

124. *To determine the direction of curvature of a given plane curve at any point, the curve being referred to rectangular co-ordinates.*

The direction of curvature of a curve at any point is the same with the direction of its tangent at that point, and is determined by the angle which such tangent makes with a given fixed line, as the axis of abscissas.

If x and y be the co-ordinates to any point of the curve, and i the angle which the tangent at that point makes with the axis of x, it has been shewn (Art. 50.) that

$$\tan i = \partial_x y;$$

$$\therefore\ i = \tan^{-1}\partial_x y.$$

Hence, if the value of $\partial_x y$ in terms of x and y, or of one of them, as x, only, be obtained from the given equation to the curve, the value of the angle i, or the direction of the curve at any point, may be found by substituting in the above equation the corresponding values of x and y, or the corresponding value of x.

Cor. Since the equation to a line passing through the point (x,y) is $Y - y = a(X - x)$, where a = tangent of the angle which the line makes with the axis of x, (Hymers' Conic Sections, Art. 20.), the equation to the tangent to a curve at the point (x,y) is

$$Y - y = \partial_x y.(X - x).$$

Ex. Required the direction of an ellipse at any point.

The equation to the ellipse referred to rectangular co-ordinates with the origin at the centre is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \ \frac{x}{a^2} + \frac{y}{b^2}.\partial_x y = 0,$$

$$\text{or} \quad \partial_x y = -\frac{b^2 x}{a^2 y},$$

$$\text{or in terms of } x = -\frac{b}{a}.\frac{x}{\sqrt{a^2 - x^2}};$$

$\therefore$ for any point (x,y)

$$i = \tan^{-1}\left(-\frac{b}{a}.\frac{x}{\sqrt{x^2 - a^2}}\right).$$

At the extremities of the axis minor, $x = 0$;

$$\therefore \ i = \tan^{-1} 0 = 0^0 \text{ or } 180^0,$$

or the direction of curvature at those points is *parallel* to the axis of x.

At the extremities of the axis major, $y = 0$, or $x = \pm a$;

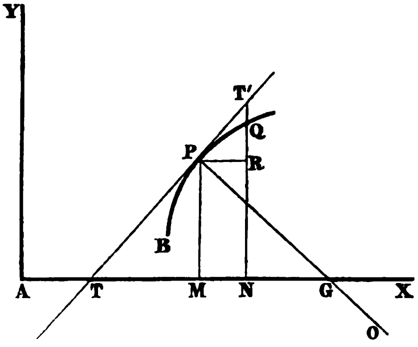
$$\therefore \ i = \tan^{-1} \infty = 90^0 \text{ or } 270^0,$$

or the direction of curvature at those points is *perpendicular* to the axis of x.

125. *To find the direction of the "normal" at any point of a plane curve referred to rectangular co-ordinates.*

Def. The normal to a curve at any point is the straight line drawn through the point at right angles to the curve, or to its tangent at that point.

Let *PT* be the tangent to the curve *BPQ* at the point *P*; *PGO* the "normal" meeting *AX* in *G*; *AM* (x), *MP* (y) the co-ordinates of *P*.



$$\text{Then } \angle T'PG = 90^\circ,$$

$$\therefore \angle PGX = 90^\circ + \angle PTG;$$

$$\therefore \tan PGX = -\cot PTG$$

$$= -\frac{1}{\tan PTG}$$

$$= -\frac{1}{\partial_x y}.$$

Hence, if the value of $\partial_x y$ in terms of x and y, or of one of them, as x, only, be obtained from the given equation to the curve, the value of the angle PGX, or the direction of the normal, for any point (x,y) of the curve may be found by substitution in the above equation.

Cor. Hence the equation to the normal at the point (x,y) is

$$Y - y = -\frac{1}{\partial_x y}(X - x).$$

126. *To find the "subtangent" for any point of a plane curve referred to rectangular co-ordinates.*

Def. The part of the axis of abscissas, intercepted between the foot of the ordinate to any point of a curve and the intersection of the tangent at that point with the axis, is called the "subtangent" for that point.

Let $AM(x)$, $MP(y)$ be the co-ordinates of the point P, (Fig. Art. 125.)

PT the tangent at P, intersecting AX in T;

then MT is the subtangent.

$$\text{And } MT = PM.\cot PTM$$

$$= \frac{PM}{\tan PTM}$$

$$= \frac{y}{\partial_x y}.$$

Hence, if the value of $\partial_x y$ in terms of x and y, or of one of them, as x, only, be obtained from the given equation to the curve, and substituted in the above expression for MT, the "subtangent" will be determined for any point (x,y) of the curve, in terms of x and y, or of one of them only.

If the value of MT thus obtained should be negative, it must be measured from M in a direction opposite to the origin of co-ordinates.

Since $\partial_x \log_e y = \frac{\partial_x y}{y} = \frac{1}{MT}$, if, in the given equation to the curve, y be an explicit function of x, the process of finding the subtangent may frequently be facilitated by taking the logarithm of each side of the equation previously to differentiation. Thus,

Ex. Required the subtangent for any point of an ellipse.

Here $$y = \frac{b}{a}\sqrt{a^2 - x^2};$$

$$\therefore \log_e y = \log_e \frac{b}{a} + \tfrac{1}{2}.\log_e \overline{a^2 - x^2};$$

$$\therefore \frac{\partial_x y}{y} = -\frac{x}{a^2 - x^2},$$

and $$\frac{y}{\partial_x y} = -\frac{a^2 - x^2}{x} = \text{subtangent for any point } (x,y).$$

Cor. The portion PT of the tangent at P intercepted between P and the axis of abscissas $= \sqrt{MP^2 + MT^2}$

$$= \sqrt{y^2 + \frac{y^2}{(\partial_x y)^2}}$$

$$= y.\sqrt{1 + (\partial_y x)^2}.$$

Also the perpendicular from the origin of co-ordinates upon the tangent at $P = AT.\frac{MP}{PT}$

$$= \left(x - \frac{y}{\partial_x y}\right).\frac{\partial_x y}{\sqrt{1 + (\partial_x y)^2}}$$

$$= \frac{x.\partial_x y - y}{\sqrt{1 + (\partial_x y)^2}}.$$

127. *To find the "subnormal" for any point of a plane curve referred to rectangular co-ordinates.*

Def. The part of the axis of abscissas, intercepted between the foot of the ordinate to any point of a curve and the intersection of the normal at that point with the axis, is called the "subnormal" for that point.

Let PM be the ordinate, (Fig. Art. 125.)
PGO the normal,
PT the tangent;
then MG is the subnormal.

$$\begin{aligned} \text{And } MG &= PM.\tan GPM \\ &= PM.\tan PTG \\ &= y.\partial_x y. \end{aligned}$$

Hence, if the value of $\partial_x y$ in terms of x and y, or of one of them as x, only be obtained from the given equation to the curve, and substituted in the above expression for MG, the "subnormal" will be determined for any point (x,y) of the curve, in terms of x and y, or of one of them only.

If the value of *MG* thus obtained should be negative, *MG* must be measured from the foot of the ordinate towards the origin of co-ordinates.

Ex. Required the subnormal for any point in an ellipse.

$$\text{Here } y^2 = \frac{b^2}{a^2}(a^2 - x^2);$$

$$\therefore\ 2y.\partial_x y = -\frac{2b^2}{a^2}x;$$

$$\therefore\ y.\partial_x y = -\frac{b^2}{a^2}x = \text{subnormal for any point } (x,y).$$

Cor. The portion *PG* of the normal intercepted between *P* and the axis of abscissas $= \sqrt{MP^2 + MG^2}$

$$= \sqrt{y^2 + y^2.(\partial_x y)^2}$$

$$= y\sqrt{1 + (\partial_x y)^2}.$$

128. *To determine whether a plane curve referred to rectangular co-ordinates is concave or convex to the axis of abscissas at any point.*

Def. A curve is said to be "concave" or "convex" to the axis of abscissas at any point according as the ordinates to the immediately preceding and succeeding points are *both* less or *both* greater than the corresponding ordinates to the tangent at that point.

Let *BPQ* (Fig. Art. 125.) be a portion of the curve;

TPT' the tangent at *P*;

AM(x), *MP*(y), the co-ordinates of *P*,

AN($x+h$), *NQ*, the co-ordinates of *Q*;

Produce *NQ*, if necessary, to meet the tangent at *P* in the point *T*; and draw *PR* parallel to *AX* meeting *NQ*, produced if necessary, in *R*.

Then the curve will be concave or convex to the axis of x at the point P, according as the immediately preceding and succeeding points in the curve lie both nearer or both farther from the axis of x than the corresponding points in the line TPT'; or according as $MP(y)$ and $NT' - NQ$ have the same or different signs, whatever be the sign of h, when h is indefinitely diminished.

Now, $\angle\, T'PR$ = angle of inclination of the tangent at P to the axis of $x = \tan^{-1} \partial_x y$;

$$\therefore\ NT' = MP + RT'$$

$$= y + \partial_x y . h.$$

And, since NQ is the same function of $x + h$ that MP is of x,

$$NQ = y + \partial_x y . \frac{h}{\underline{|1}} + \partial_x^2 y . \frac{h^2}{\underline{|2}} + \&c. \quad \text{(Art. 100.)}$$

$$\therefore\ NT' - NQ = -\partial_x^2 y . \frac{h^2}{\underline{|2}} - \partial_x^3 y . \frac{h^3}{\underline{|3}} - \&c.$$

But, by sufficiently diminishing h, the sign of the latter series may be made the same with the sign of its first term, (Art. 100. Cor.) which sign will not change with the sign of h;

$\therefore$ $NT' - NQ$ and $-\partial_x^2 y$ have the same signs, whatever be the sign of h, when h is indefinitely diminished.

Hence the curve will be concave or convex at the point (x, y) according as

y and $-\partial_x^2 y$ have the same or different signs;

or according as

y and $\partial_x^2 y$ have different or the same signs;

or as

$y . \partial_x^2 y$ is negative or positive.

If by the substitution of the given values of x and y, $\partial_x^2 y$ and also $\partial_x^3 y$ should be rendered equal to 0, the curve will in like manner be concave or convex according as $y.\partial_x^4 y$ is negative or positive;—and generally, if the substitution of any proposed values of x and y render $\partial_x^2 y$, $\partial_x^3 y$,......$\partial_x^{2n-1} y$ separately equal to 0, the curve will be concave or convex to the axis of x at the point corresponding to those values, according as $y.\partial_x^{2n} y$ is negative or positive.

Ex. Given the equation to a circle $(x-a)^2+(y-b)^2=r^2$, to determine what portion of the curve is concave or convex to the axis of x.

$$(x-a)^2+(y-b)^2=r^2;$$

$$\therefore\ x-a+(y-b).\partial_x y=0,$$

$$\text{and}\ \ 1+(\partial_x y)^2+(y-b).\partial_x^2 y=0;$$

$$\therefore\ 1+\frac{(x-a)^2}{(y-b)^2}+(y-b).\partial_x^2 y=0,$$

$$\text{or}\ \ \partial_x^2 y=-\frac{r^2}{(y-b)^3};$$

$$\therefore\ y.\partial_x^2 y=-\frac{y}{y-b}.\left(\frac{r}{y-b}\right)^2.$$

Hence, since $\left(\frac{r}{y-b}\right)^2$ is necessarily positive, if b be supposed positive and $> r$, in which case y is always positive, $y.\partial_x^2 y$ will be negative or positive, and the curve will be concave or convex, according as y is $>$ or $<$ b; or according as the point is taken above or below that diameter of the circle which is parallel to the axis of x.

129. If for the given value of x, as a, $y=0$; or if one or both of the values of y corresponding to $x=a\pm h$, when h is indefinitely diminished, be impossible; or if $(\partial_x y)_{x=a}=\infty$, i.e. the tangent to the curve at the given point be perpendicular to the axis of x; it is evident that the preceding reasoning is inapplicable, and that the curve cannot at the given point be either "concave" or "convex" to the axis

of x, except in the case when it *touches* that axis, and the contiguous points of the branches in opposite directions from the point of contact lie on the same side of the axis, in which case the curve will be "convex" to the axis of x.

130. *To find whether a curve which touches the axis of abscissas is* "convex" *to that axis at the point of contact.*

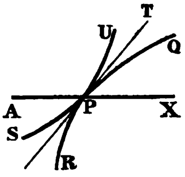
Let $x = a$ be the abscissa corresponding to the point of contact; then, since $y_{x=a} = 0$, and $(\partial_x y)_{x=a} = 0$, the values of $y_{x=a\pm h}$ are

$$(\partial_x^2 y)_{x=a}\cdot\frac{h^2}{\lfloor 2} \pm (\partial_x^3 y)_{x=a}\cdot\frac{h^3}{\lfloor 3} + (\partial_x^4 y)_{x=a}\cdot\frac{h^4}{\lfloor 4} \pm \&c.;$$

and, since when h is sufficiently diminished, the sign of these series will be the same with the signs of their first terms, it is evident that, when h is indefinitely diminished, the values of $y_{x=a\pm h}$ will have the same sign—or the contiguous points of the branches in opposite directions from the point of contact will lie on the same side of the axis of x, i. e. *the curve will be "convex"—when the first differential of y, which does not become equal to* 0 *for the given value of x, is of an even order.* And it is also evident that the branches of the curve will lie both above or both below the axis, according as the sign of such differential is positive or negative.

131. But although, at a point where the curve cuts the axis of x—or where one of the values of $y_{x=a\pm h}$, when h is indefinitely diminished, is impossible—or where the tangent to the curve is perpendicular to the axis of x—the curve cannot be "concave" or "convex" to the axis of x within the meaning of the term as defined in Art. 128: it may nevertheless be determined whether any branch of the curve at the proposed point, considered as proceeding from that point, is concave or convex, i. e. lies with regard to the axis of x in a manner similar to the branches *PQ*, *PS*, or to the branches *PR*, *PU* in the annexed figs.:—

(1)



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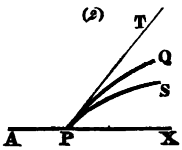

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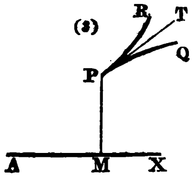

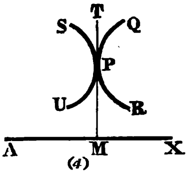

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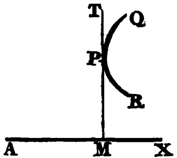

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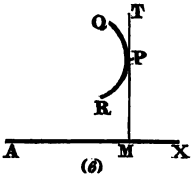
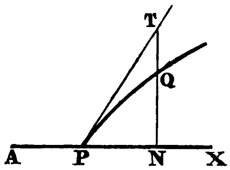


(6)

132. ***When a curve cuts the axis of abscissas at a finite oblique angle, to find whether any branch of the curve, considered as proceeding from the point of intersection, is concave or convex to the axis at that point.***



Let PQ be the proposed branch of the curve;

$AP (x)$,

$AN (x+h)$, NQ, the co-ordinates of Q.

And let NQ, produced if necessary, intersect the tangent at P in the point T.

Then it is evident that the branch PQ will be concave or convex to the axis of x according as Q does or does not fall between N and T, or according as $NT - NQ$ is positive or negative, when h is indefinitely diminished.

But $NT - NQ = -\{\partial_x^2 y.\frac{h^2}{\underline{\lfloor 2}} + \partial_x^3 y.\frac{h^3}{\underline{\lfloor 3}} + \&c.\}$ Art. 128.

and, when h is sufficiently diminished, the sign of this series will be the same with the sign of the first term of which the coefficient is not equal to 0; therefore, since h is not supposed to change its sign, $NT - NQ$ will be positive or negative when h is indefinitely diminished,—i.e. *the branch will be concave or convex,—according as the first of the differentials* $\partial_x^2 y$, $\partial_x^3 y$, &c., *which does not become equal to* 0 *for the given value of* x, *is negative or positive.*

If $-h$ be substituted for h in the above series the concavity or convexity of the corresponding branch may in like manner be determined, and the branch will be concave or convex according as the first term of the series

$$\partial_x^2 y . \frac{h^2}{\underline{|2}} - \partial_x^3 y . \frac{h^3}{\underline{|3}} + \partial_x^4 y . \frac{h^4}{\underline{|4}} - \text{\&c.}$$

which does not become equal to 0, independently of h, is negative or positive.

If the first term which does not become equal to 0 by the substitution of the given value of x should have more than one value, the concavity or convexity of the branch corresponding to each of such values may in like manner be determined.

133. It is evident that the preceding reasoning is applicable to any branch of a curve considered as proceeding from a given point in it, if the tangent to the curve at that point be oblique to the axis of x, whether part of the curve lies on each side of the given point, or the whole lies on one side, as in Figs. (2) and (3), Art. 131; except that in the latter case, as there is no value of NQ corresponding to *both* the values of x, $a + h$ and $a - h$, when h is indefinitely diminished, the series for NQ in powers of h must contain one or more terms in which the index of h is of the form $\frac{2m+1}{2n}$, and therefore cannot be obtained by Taylor's Theorem. (Vide Art. 102.)

134. If there should be no possible value of $y_{x=a\pm h}$ when h is indefinitely diminished, there is no part of the curve immediately contiguous to the proposed point, and

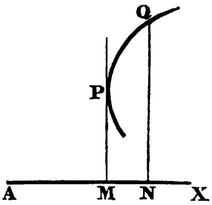
therefore the curve cannot be said to be either concave or convex at that point. Points of this sort are called "isolated points," and will be treated of hereafter.

135. *To find whether any branch of a curve, considered as proceeding from a point at which the tangent is perpendicular to the axis of abscissas, is concave or convex to the axis at that point.*

Let PQ be the proposed branch of the curve;

$AM(x)$, $MP(y)$, the co-ordinates of P;

$AN(x+h)$, NQ .... Q.



Then MP is the tangent to the curve at P, and therefore, it is evident that the branch PQ will be concave or convex to the axis of x at the point P, according as NQ is > or < MP independently of its sign, when h is indefinitely diminished.

Let the value of NQ in a series of ascending positive powers of h, if capable of being so expressed, be represented by

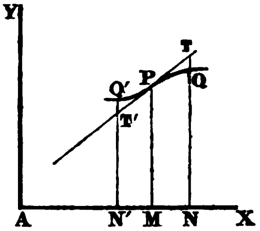
$$y + R.h^{\alpha} + S.h^{\beta} + \&c.$$

$$\text{Then } NQ - MP = R.h^{\alpha} + S.h^{\beta} + \&c.;$$

and since, when h is sufficiently diminished, the sign of this series is the same with the sign of its first term, $NQ - MP$ will be positive or negative, when h is indefinitely diminished —or *the branch will be concave or convex,—according as* Rh^{α} *is positive or negative.*

136. *To find the points of "contra-flexure," if there be any, of a plane curve referred to rectangular co-ordinates.*

Def. A point of "contra-flexure" is one for which the immediately preceding and succeeding points of the curve lie on different sides of the tangent, in the manner indicated in the annexed fig.



Let QPQ' be a portion of a plane curve having a point of contra-flexure at P;

TPT' the tangent at P;

$AM(x)$, $MP(y)$, the co-ordinates of P,

$AN(x+h)$, NQ, the co-ordinates of Q;

$AN'(x-h)$, $N'Q'$, the co-ordinates of Q'.

Produce NQ, $N'Q'$ if necessary to meet the tangent at P in the points T and T'.

Then it is evident, from the definition, that

$$NT - NQ \text{ and } NT' - N'Q',$$

$$\text{or} \quad -\partial_x^2 y.\frac{(\pm h)^2}{\underline{|2}} - \partial_x^3 y.\frac{(\pm h)^3}{\underline{|3}} - \&c. \quad \text{(Art. 128.)}$$

will have different signs, when h is indefinitely diminished.

But since, by sufficiently diminishing h, the sign of each series may be made the same with that of its first term, this condition cannot be satisfied, unless the first term is equal to 0. Hence the values of x and y corresponding to points of contra-flexure, must satisfy the equation

$$\partial_x^2 y = 0,$$

and also the original equation to the curve.

If any of these values of x and y, when substituted, render $\partial_x^3 y$ positive, the curve will be concave to the axis of x for the branch corresponding to $+h$, as PQ, and convex for that corresponding to $-h$, as PQ'; and *vice versâ*, if $\partial_x^3 y$ be rendered negative.

If any such values of x and y render $\partial_x^3 y$ equal to 0, the corresponding points will not be points of contra-flexure unless the same values render $\partial_x^4 y$ also equal to 0; and generally, the points corresponding to any values of x and y will not be points of contra-flexure, unless the first differential after $\partial_x^2 y$ which remains finite for those values is of an odd order.

137. The conclusions obtained in the last article would also be arrived at by considering that the angle of inclination of the tangent to the axis of abscissas must be greater or less at a point of contra-flexure than at both the immediately preceding and succeeding points of the curve, and consequently that at such a point the trigonometrical tangent of that angle, or $\partial_x y$, must be a maximum or minimum, i. e. $\partial_x(\partial_x y)$, or $\partial_x^2 y$, must be equal to 0; or more generally, the first differential, which remains finite on the substitution of the corresponding values of x and y, must be of an odd order.

Ex. 1. Required the points of contra-flexure in the curve whose equation is $a^{n-1}y = x^n$.

$$a^{n-1}y = x^n;$$

$$\therefore\ a^{n-1}.\partial_x y = nx^{n-1},$$

$$a^{n-1}.\partial_x^2 y = n.\overline{n-1}.x^{n-1}.$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$a^{n-1}.\partial_x^n y = \underline{|n}.$$

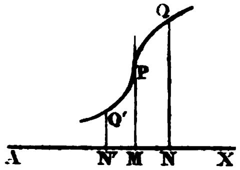
Now $\partial_x^2 y = 0$ gives $x = 0$, and the first differential which remains finite when this value of x is substituted in it, is of an odd or even order according as n is odd or even; therefore the proposed curve has a point of contra-flexure or not at the origin, according as n is odd or even.

138. The construction and reasoning in the two preceding Articles are clearly inapplicable to points of the curve at which the tangent is perpendicular to the axis of x, i. e. where $\partial_x y = \infty$; but since a tangent to the curve, which is perpendicular to the axis of x, is parallel to the axis of y, the points of contra-flexure, if there be any, at which the tangent is perpendicular to the axis of x, may be found by taking the differentials of x with regard to y, instead of those of y with regard to x, and proceeding as in Art. 136.

The points of contra-flexure at which the curve is perpendicular to the axis of x may, however, frequently be found more readily by the following method:—

139. *To find the points of contra-flexure, if there be any, of a plane curve referred to rectangular co-ordinates, where the tangent to the curve is perpendicular to the axis of abscissas.*

Let P be a point in the curve QPQ' at which the tangent is perpendicular to AX.



$AM(x)$, $MP(y)$, co-ordinates of P,

$AN(x+h)$, NQ .... Q,

$AN'(x-h)$, NQ' .... Q',

Then, it is evident that, if P be a point of contra-flexure, one of the quantities NQ, $N'Q'$ will be greater, and the other less than, MP.

Let the value of $NQ - MP$ expanded in positive ascending powers of h, when it is capable of such expansion, be represented by

$$R.h^{\alpha} + S.h^{\beta} + \&c.$$

then the value of $N'Q' - MP$ will be

$$R.(-h)^{\alpha} + S.(-h)^{\beta} + \&c.;$$

and since, when h is sufficiently diminished, the sign of each of these series will be the same with the sign of its first term,

$NQ - MP$ and $N'Q' - MP$ will have different signs, or P will be a point of contra-flexure, when h^{α} and $(-h)^{\alpha}$ have different signs, i. e. when α is of the form $\dfrac{2m+1}{2n+1}$*.

Ex. Required the points of contra-flexure in the curve whose equation is $ax^2 = y^3 + a^3$.

By taking the differentials of y with regard to x, and proceeding according to Art. 136, it will be found that there is no possible value of x which will satisfy the equation $\partial_x^{2n} y = 0$; and therefore no points of contra-flexure can be detected by this method.

---

* α cannot be an integral number, since in that case R would be one of the differentials of y, all of which are separately equal to ∞. (Art. 102.)

Again, taking the differentials of x with regard to y,

$$2\,ax.\partial_y x = 3\,y^2,$$

$$\text{and}\quad 2\,a.(\partial_y x)^2 + 2\,ax.\partial_y^2 x = 6\,y\,;$$

$$\therefore\quad 2\,ax.\partial_y^2 x = 6y - 2\,a.\frac{9\,y^4}{4\,a^2 x^2}$$

$$= 6\,y - \frac{9\,y^4}{2\,(y^3 + a^3)}$$

$$= \frac{3\,y^4 - 12\,a^3 y}{2\,(y^3 + a^3)}\,;$$

$$\text{and}\quad \partial_y^2 x = \frac{3\,y^4 - 12\,a^3 y}{4\,a^{\frac{1}{2}}.(y^3 + a^3)^{\frac{3}{2}}}.$$

Hence it appears that

$$\left.\begin{matrix} y = 0 \\ x = a \end{matrix}\right\} \quad\text{and}\quad \left.\begin{matrix} y = 0 \\ x = -\,a \end{matrix}\right\}$$

satisfy the equation $\partial_y^2 x = 0$, and the original equation to the curve; and it will be found on trial that these values, when substituted, render $\partial_y^3 x$ finite; therefore the two points of the curve corresponding to these co-ordinates are points of contra-flexure. It appears also that at these points the curve is perpendicular to the axis of x.

The points of contra-flexure in the above example may however be found more easily by the method pointed out in Art. 139:—

$$\text{For, since}\quad ax^2 = y^3 + a^3,$$

$$2\,ax.\partial_y x = 3\,y^2;$$

and since the values $y = 0$, $x = \pm\,a$, evidently satisfy the original equation, and render $\partial_y x = 0$ or $\partial_x y = \infty$, the tangent to the curve is perpendicular to the axis of x at the points of the curve corresponding to $y = 0$, $x = a$, and $y = 0$, $x = -\,a$.

But $y^3 = ax^2 - a^3$; and if y' be the value of y corresponding to the abscissa $a + h$,

$$y'^3 = a(a+h)^2 - a^3$$

$$= 2a^2h + ah^2;$$

$$\therefore\ y' = h^{\frac{1}{3}}(2a^2)^{\frac{1}{3}}\left\{1 + \frac{h}{2a}\right\}^{\frac{1}{3}},$$

and since $y = 0$, when $x = a$,

$$y' - y = h^{\frac{1}{3}}(2a^2)^{\frac{1}{3}}.\left\{1 + \tfrac{1}{3}.\frac{h}{2a} + \&c.\right\}$$

in which the index of the first power of h is $\frac{1}{3}$, i.e. of the form $\dfrac{2m+1}{2n+1}$.

Therefore the curve has a point of contra-flexure corresponding to $x = a$, $y = 0$, and in the same manner it may be shewn that it has such a point also corresponding to $x = -a$, $y = 0$.

140. *To find the points, if there be any, in a plane curve referred to rectangular co-ordinates, at which the tangent is perpendicular or parallel to the axis of abscissas.*

Let $u = f(x,y) = 0$ be the equation to the curve; then

$$\partial_x y = -\frac{\partial_{(x)}u}{\partial_{(y)}u}.\quad \text{Art. 74.}$$

I. Now for those points in the curve at which the tangent is perpendicular to the axis of x, $\partial_x y = \infty$,

$$\therefore\ \partial_{(y)}u = 0.$$

Hence the values of x and y corresponding to such points must satisfy the equations

$$u = 0,\quad \partial_{(y)}u = 0,$$

and not render $\partial_{(x)}u$ equal to 0.

II. Again, for those points in the curve at which the tangent is parallel to the axis of x, $\partial_x y = 0$,

$$\therefore\ \partial_{(x)}u = 0;$$

and the values of x and y corresponding to such points must satisfy the equations

$$u = 0, \quad \partial_{(x)} u = 0,$$

and not render $\partial_{(y)} u$ equal to 0.

It is evident that, in the latter case, if the values of x and y render $\partial_x^2 y$ a finite quantity, the *ordinate* will be a "maximum" or a "minimum." (Vide Art. 114.) Also, in the former case, if the values of x and y render $\partial_y^2 x$ a finite quantity, the *abscissa* will be a "maximum" or "minimum."

Ex. Required the points at which the tangent to the curve, whose equation is $x^3 - 3axy + y^3 = 0$, is perpendicular or parallel to the axis of x.

$$u = x^3 - 3axy + y^3;$$

$$\therefore \; \partial_{(x)} u = 3x^2 - 3ay,$$

$$\partial_{(y)} u = 3y^2 - 3ax;$$

$\therefore$ the tangent to the curve is perpendicular to the axis of x for those values of x and y which satisfy the equations

$$x^3 - 3axy + y^3 = 0, \text{ and } y^2 - ax = 0;$$

which values are $x = \sqrt[3]{4}.a, \quad y = \sqrt[3]{2}.a.$

Also the tangent to the curve is parallel to the axis of x for those values of x and y which satisfy the equations

$$x^3 - 3axy + y^3 = 0, \text{ and } x^2 - ay = 0;$$

which values are $x = \sqrt[3]{2}.a, \quad y = \sqrt[3]{4}.a.$

141. *To determine the "multiple" "osculating," and "isolated" points, if there be any, in a plane curve referred to rectangular co-ordinates.*

Def. A "multiple" point is one at which two or more branches of the curve intersect, making finite angles with each other.

An "osculating" point is one at which two or more branches of the curve meet, and have a common tangent.

An "isolated" point is one which is detached from the rest of the curve*.

Let $u = f(x, y) = 0$ be the equation to the curve, cleared of radicals; then (Art. 74.)

$$\partial_{(x)}u + \partial_{(y)}u . \partial_x y = 0, \dots\dots\dots\dots (1)$$

$$\partial^2_{(x)}u + 2.\partial_{(x)}\partial_{(y)}u.\partial_x y + \partial^2_{(y)}u.(\partial_x y)^2 + \partial_{(y)}u.\partial^2_x y = 0,$$

And generally, taking the n^{th} differential of u,

$$\partial^n_{(x)}u + n.\partial^{n-1}_{(x)}\partial_{(y)}u.\partial_x y + \frac{n(n-1)}{2}.\partial^{n-2}_{(x)}\partial^2_{(y)}u.(\partial_x y)^2 + \dots + \partial_{(y)}u.\partial^n_x y = 0 \dots (2).$$

I. Now for the values of x and y corresponding to a "*multiple*" point, $\partial_x y$ will have several values, while $\partial_{(x)}u$ and $\partial_{(y)}u$ have respectively but one value. Hence equation (1) must be satisfied independently of $\partial_x y$.

$$\therefore \quad \partial_{(x)}u = 0, \quad \text{and} \quad \partial_{(y)}u = 0.$$

II. Again, since the ordinate corresponding to the abscissa $x + h$, when h is sufficiently diminished, is equal to

$$y + \partial_x y.h + \partial^2_x y.\frac{h^2}{\underline{|2}} + \partial^3_x y.\frac{h^3}{\underline{|3}} + \&c.$$

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[*] A point of this nature is frequently called a "*conjugate*" point, on the supposition that it is produced by the *coalescing* into a point of some re-entering curve, which belongs to the equation in a more general form. Thus the equation to two concentric circles, whose centres coincide with the origin of co-ordinates and whose respective radii are a and b, when solved with regard to y, is

$$y = \tfrac{1}{2}\left\{\sqrt{a^2 - x^2} + \sqrt{b^2 - x^2} \pm \left(\sqrt{a^2 - x^2} - \sqrt{b^2 - x^2}\right)\right\},$$

and if 0 be substituted for a, by which one of the circles is reduced to a point, the corresponding equation

$$y = \tfrac{1}{2}\left\{\sqrt{-x^2} + \sqrt{b^2 - x^2} \pm \left(\sqrt{-x^2} - \sqrt{b^2 - x^2}\right)\right\},$$

is the equation to a curve consisting of a circle, whose radius is b, and a "*conjugate*" point at its centre.

it is evident that, for the values of x and y corresponding to an "*osculating*" point, $\partial_x^2 y$, or some higher differential, as $\partial_x^n y$, will have more than one value, while y, $\partial_x y$, and any other lower differential have respectively but one value. Hence for such values of x and y equation (2) must be satisfied independently of $\partial_x^n y$;

$$\therefore\ \partial_{(y)} u = 0,$$

and therefore, from equation (1), if $\partial_x y$ be not ∞,

$$\partial_{(x)} u = 0.$$

If $\partial_x y$ be ∞, $\partial_y x = 0$ for the same values of x and y, and therefore, if the differentials be taken with regard to y instead of x, it will appear as before that

$$\partial_{(x)} u = 0, \quad \text{and } \partial_{(y)} u = 0.$$

III. Again, for the values of x and y corresponding to an "*isolated*" point, since the ordinate corresponding to $x+h$ is impossible when h is sufficiently diminished, it is evident that some term in the series

$$y + \partial_x y.h + \partial_x^2 y.\frac{h^2}{\underline{|2}} + \partial_x^3 y.\frac{h^3}{\underline{|3}} + \&c.$$

must be impossible; and, therefore, if $\partial_x^n y.\frac{h^n}{\underline{|n}}$ be the first such term, equation (2) must be satisfied independently of $\partial_x^n y$;

$$\text{Hence, } \partial_{(y)} u = 0, \quad \text{and } \therefore\ \partial_{(x)} u = 0.$$

Therefore the values of x and y corresponding to "*multiple*," or "*osculating*" or "*isolated*" points must in every case satisfy the three equations,

$$u = 0, \quad \partial_{(x)} u = 0, \quad \partial_{(y)} u = 0;$$

a condition by which the *positions* of all such points in a curve may be determined.

But further, to determine the *nature* of the point corresponding to any such values of x and y.

Since, for all such values

$$\partial_x y = \left\{-\frac{\partial_{(x)} u}{\partial_{(y)} u}\right\} = \frac{0}{0};$$

it may be shewn as in Art. 97, that for the same values

$$\partial^n_{(x)} u + n.\partial^{n-1}_{(x)} \partial_{(y)} u.\partial_x y + \ldots + n.\partial_{(x)} \partial^{n-1}_{(y)} u.(\partial_x y)^{n-1} + \partial^n_{(y)} u.(\partial_x y)^n = 0,$$

(n being such a number that this equation is the first of that form which is not satisfied by the coefficients of the powers of $\partial_x y$); and if for such values of x and y no higher differential be impossible and this equation have any number of *unequal* possible roots, the point will be a "*multiple*" point, at which there are a like number of tangents whose positions will be determined by the corresponding values of $\partial_x y$.

If it have any number of *equal* possible roots, the point will be an "*osculating*" point, at which a like number of branches of the curve will have a common tangent.

If it have both unequal and equal possible roots the point will be both "*multiple*" and "*osculating*."

If all the roots of the equation be impossible, or if any higher differential of y be impossible, the point will be an "*isolated*" point.

Ex. Required the "multiple," "osculating," or "isolated" points in the curve whose equation is $x^4 - ayx^2 + by^3 = 0$.

$$\text{If} \quad u = x^4 - ayx^2 + by^3 = 0,$$

$$\partial_{(x)} u = 4x^3 - 2ayx = 0,$$

$$\partial_{(y)} u = 3by^2 - ax^2 = 0;$$

which three equations are severally satisfied by the values

$$x = 0, \quad \text{and} \quad y = 0.$$

Therefore, proceeding as in Art. 97. Ex., it appears that the origin is a "multiple" point where *three* branches meet the axis of x, one of which touches it, and the other two intersect at angles $\tan^{-1}\left(\pm\sqrt{\frac{a}{b}}\right)$, a and b being supposed to represent positive quantities.

142. If the given equation to the curve be solved with respect to one of the co-ordinates, as y, the "multiple", "osculating", or "isolated", points may often be detected more easily than by the previous method. For, if the value of y in terms of x contain a radical affected by a coefficient which becomes equal to 0 when a particular value is given to x, the point of the curve corresponding to such a value is frequently "multiple", "osculating", or "isolated"; and the nature of it may be determined by taking some of the successive differentials of y and substituting in them the particular value of x, as in the following Examples:

Ex. 1. $y^2 = x.(a-x)^2$.

$$\text{Here } y = \pm\sqrt{x}.(a-x),$$

$$\text{and } \partial_x y = \pm (a-x).\frac{1}{2\sqrt{x}} \mp \sqrt{x}.$$

$$\text{Now } y_{x=a} = 0, \quad \text{and } (\partial_x y)_{x=a} = \pm\sqrt{a};$$

therefore the point corresponding to $x=a$, $y=0$, is a "multiple" point at which the tangents are inclined to the axis of x at angles $\tan^{-1}\sqrt{a}$ and $\tan^{-1}(-\sqrt{a})$.

Ex. 2. $y = \pm (a-x)^2.\sqrt{x}$.

$$\text{Here } \partial_x y = \pm (a-x)^2.\frac{1}{2\sqrt{x}} \mp 2(a-x).\sqrt{x}$$

$$= \pm\frac{\overline{a-x}.\overline{a-3x}}{2\sqrt{x}}.$$

$$\partial_x^2 y = \pm\frac{1}{4x}\left\{-2\sqrt{x}.(3.\overline{a-x}+a-3x) - \overline{a-x}.\overline{a-3x}.x^{\frac{1}{2}}\right\}.$$

$$\text{Now } y_{x=a} = 0,$$

$$\text{and } (\partial_x y)_{x=a} = 0,$$

$$\text{but } (\partial_x^2 y)_{x=a} = \pm\sqrt{a};$$

∴ the point corresponding to $x=a$, and $y=0$, is an "osculating" point of which the common tangent to the two branches of the curve is parallel to the axis of x.

Ex. 3. $y^2 = b^2 \pm (x^2 - a^2).\sqrt{a^2 - 2x^2}$.

$$\text{Here } y.\partial_x y = \pm 2x.\sqrt{a^2 - 2x^2} \mp \frac{(x^2 - a^2).2x.}{\sqrt{a^2 - 2x^2}}$$

$$= \frac{\pm 2x}{\sqrt{a^2 - 2x^2}} \{2a^2 - 3x^2\}.$$

$$\text{Now, let } x = \pm a, \text{ then } y = \pm b;$$

$$\text{but } (\partial_x y)_{x=\pm a} \text{ is impossible};$$

∴ the points corresponding to $x = \pm a$, $y = \pm b$ are "isolated" points.

143. *To determine the "cusps", if there be any, in a plane curve referred to rectangular co-ordinates.*

Def. If two branches of a curve proceed from a point P, at which they have a common tangent, in the manner indicated in Figs. 2, 3, 4, Art. 131, the point P is called a cusp.

The "cusp" is said to be of the "first species" when the branches lie on opposite sides of the common tangent, as in Figs. 3 and 4:—of the "second species" when on the same side, as in Fig. 2.

It appears from the above Def., that if a be a value of x for which (the tangent to the curve not being perpendicular to the axis of x,) $y_{x=a}$, and $(\partial_x y)_{x=a}$ have respectively but *one* value, while one of the quantities $y_{x=a\pm h}$ has *two* values and the other is impossible when h is indefinitely diminished, the corresponding point of the curve is a "cusp":—and that, if the tangent be perpendicular to the axis of x, b be the corresponding value of y, and one of the quantities $x_{y=b\pm h}$ has *two* values while the other is impossible when h is indefinitely diminished, then also the point is a "cusp".

To find the values of a and b in any proposed case, solve the given equation with regard to y; then, if the resulting equation contain a single function of x of the form $P^{\frac{2m+1}{2n}}$, the corresponding term will be affected by the double sign $\pm$; and if a be a root of the equation $P = 0$, and the substitution of one of the quantities $a \pm h$ for x render P negative but the coefficient of P in the equation not equal to 0, it is evident that the corresponding point will be a "cusp", provided $(\partial_x y)_{x=a}$ have but one value, and such value be not ∞.

If $(\partial_x y)_{x=a} = \infty$, i.e., the tangent be perpendicular to the axis of x, the corresponding point may be either a cusp, as P Fig. 4. Art. 131, or a point situated as P Figs. 5 and 6. In the former case, the point P in the curve *UPR* or *SPQ* will not be discoverable by the preceding method; but may be determined as follows:—

Solve the equation with regard to x; then, if the resulting equation contain a function of y of the form $Q^{\frac{2m+1}{2n}}$, such that $Q_{y=b} = 0$, and one of the quantities $Q_{y=b\pm h}$ is negative, and $(\partial_y x)_{y=b} = 0$, the corresponding point is a cusp.

To determine the "species" of the cusp, that value of $y_{x=a\pm h}$, or $x_{y=b\pm h}$, which is not impossible, must be obtained in a series of powers of h, and the concavity or convexity of the corresponding branches of the curve determined as in Arts. 132, 133, 135.

If the point be not on the axis, and one branch be concave and the other convex, the cusp is one of the "first species"; but if both branches be concave, or convex, the cusp is one of the "second species".

If the point be on the axis, and the branches lie on the opposite sides of the axis the cusp is of the "first species", if on the same side the cusp is of the "second species".

Ex. Required the "cusp" belonging to the curve whose equation is

$$(y - b - cx^2)^2 = (x - a)^5.$$

Solving the equation with regard to y,

$$y = b + cx^2 \pm (x - a)^{\frac{5}{2}},$$

which contains a function of x, viz. $(x-a)^{\frac{5}{2}}$, of the form $P^{\frac{2m+1}{2n}}$, and a is a root of $P = 0$, or $x - a = 0$.

Now when $a-h$ is substituted in P for x, P is negative, while its coefficient is finite; also $(\partial_x y)_{x=a} = 2ca$ has but one value.

Hence $x=a$, $y=b+ca^2$, are the co-ordinates to a "cusp".

$$\text{Again, } y_{x=a+h} = b + ca^2 + 2ach + ch^2 \pm h^{\frac{5}{2}};$$

$$\therefore\ y_{x=a+h} - y_{x=a} = 2ach + ch^2 \pm h^{\frac{5}{2}},$$

both of which values are positive when h is sufficiently diminished. Hence the cusp is of the "second species".

II. PROPERTIES OF PLANE CURVES WITH RELATION TO OTHER CURVES.

CONTACTS, CIRCLE OF CURVATURE, AND EVOLUTES.

144. Let two plane curves be referred to the same axes of rectangular co-ordinates, and Y, y be the ordinates for a common abscissa x, and $Y=Fx$, $y=fx$ the equations to the curves; then

$$F\overline{x+h} - f\overline{x+h} = Y - y + (\partial_x Y - \partial_x y).h + (\partial_x^2 Y - \partial_x^2 y).\frac{h^2}{\lfloor 2} + \text{\&c.}\dots$$

If now $Y=y$, the two curves meet at the point (x,y).

If also $\partial_x Y = \partial_x y$, ... have a contact which is said to be of the first order.

...... $\partial_x^2 Y = \partial_x^2 y$, ... have a contact which is said to be of the second order.

And generally, if such equations hold as far as $\partial_x^n Y = \partial_x^n y$, the two curves have a contact which is said to be of the n^{th} order at the point corresponding to abscissa x.

It is obvious that the above series expresses the distance between the curves, measured along the ordinate corresponding to a common abscissa $\overline{x+h}$, which, when h is indefinitely diminished, is less or greater, according as the order of contact is higher or lower.

145. If one of the curves be given, the other may be made to have with it a contact of the n^{th} order, provided that the equation of the latter contain $\overline{n+1}$ arbitrary constants. For these constants may be so assumed as to satisfy the $\overline{n+1}$ equations

$$Y=y,\quad \partial_x Y=\partial_x y,\quad \partial_x^2 Y=\partial_x^2 y, \ldots\ldots \partial_x^n Y=\partial_x^n y.$$

Thus the equation to a straight line, being $Y=ax+b$, contains two arbitrary constants, which may be so assumed as to satisfy the two equations $Y=y$, $\partial_x Y=\partial_x y$. Hence a straight line may be made to have a contact of the *first order* with a given curve at any point, in which case it coincides with the tangent to the curve at that point, (Art. 50.); and if X, Y be co-ordinates of any point in the straight line which is a tangent to the curve at the point (x,y), the equation to it is

$$Y-y=\partial_x y.(X-x).$$

Again, the equation to a circle, being $(x-\alpha)^2+(Y-\beta)^2=R^2$, contains *three* arbitrary constants, which may be so assumed as to satisfy the *three* equations $Y=y$, $\partial_x Y=\partial_x y$, $\partial_x^2 Y=\partial_x^2 y$. Hence a circle may be made to have a contact of the *second order* with a given curve at any point; and the circle which has a contact of the second, or any higher order, with the curve at any point is called the "circle of curvature" to the curve at that point.

146. *To determine in what cases the tangent, or the circle of curvature,* CUTS *the curve at the point of contact.*

Let $y=fx$ represent the equation to the curve,

$Y=Fx$..................................tangent or circle.

Then $$F\overline{x+h}-f\overline{x+h}=Y-y+(\partial_x Y-\partial_x y).\frac{h}{1}+(\partial_x^2 Y-\partial_x^2 y)\frac{h^2}{\underline{|2}}+\&c.$$

I. For the tangent, $Y=y$, $\partial_x Y=\partial_x y$, and $\partial_x^2 Y=\partial_x^3 Y=\&c.=0.$

$$\therefore\ F\overline{x+h}-f\overline{x+h}=-\partial_x^2 y.\frac{h^2}{\underline{|2}}-\partial_x^3 y.\frac{h^3}{\underline{|3}}-\&c.$$

which changes its sign with the sign of h, if h be sufficiently diminished, when the first differential of y which does not become equal to 0 is of an odd order, or the tangent has with the curve a contact of the second or some higher even order. Hence the tangent *cuts* the curve at the point of contact only when that point is a point of "contraflexure". (Art. 136.)

II. For the circle $Y = y$, $\partial_x Y = \partial_x y$, and $\partial_x^2 Y = \partial_x^2 y$;

$$\therefore\ F\overline{x+h} - f\overline{x+h} = (\partial_x^3 Y - \partial_x^3 y).\frac{h^3}{\lfloor 3} + \&c.$$

which changes its sign with the sign of h, if h be sufficiently diminished, when the index of the first power of h in the series whose coefficient does not become equal to 0 is odd, or the circle has with the curve a contact of the third or some higher odd order. Hence, as the "circle of curvature" has usually a contact of the second order only with the curve, such circle generally *cuts* the curve at the point of contact.

147. *To determine the circle of curvature at any proposed point of a given curve referred to rectangular co-ordinates.*

Assume α and β to be the co-ordinates of the centre, and R the radius, of the circle of curvature at the point (x,y) of the curve.

$$\text{Then } (X-\alpha)^2 + (Y-\beta)^2 = R^2.$$

$$\text{Also } X - \alpha + (Y-\beta).\partial_X Y = 0,$$

$$\text{and } 1 + (\partial_X Y)^2 + (Y-\beta).\partial_X^2 Y = 0,$$

hold for all corresponding values of X, Y, $\partial_X Y$, and $\partial_X^2 Y$.

But at the point (x,y) of contact with the curve,

$$X = x,\ Y = y,\ \partial_X Y = \partial_x y,\ \partial_X^2 Y = \partial_x^2 y;$$

$$\therefore\ (x-\alpha)^2 + (y-\beta)^2 = R^2,$$

$$x - \alpha + (y-\beta).\partial_x y = 0,$$

$$1 + (\partial_x y)^2 + (y-\beta).\partial_x^2 y = 0.$$

Hence, $y - \beta = -\dfrac{1}{\partial_x^2 y}\left\{1 + (\partial_x y)^2\right\}$,

and $x - \alpha = \dfrac{\partial_x y}{\partial_x^2 y}\left\{1 + (\partial_x y)^2\right\}$;

$$\therefore\ \alpha = x - \frac{\partial_x y}{\partial_x^2 y}\left\{1 + (\partial_x y)^2\right\},$$

and $\beta = y + \dfrac{1}{\partial_x^2 y}\left\{1 + (\partial_x y)^2\right\}$.

Also $R^2 = \left\{1 + (\partial_x y)^2\right\}^2 . \dfrac{1 + (\partial_x y)^2}{(-\partial_x^2 y)^2}$;

$$\therefore\ R = \frac{\left\{1 + (\partial_x y)^2\right\}^{\frac{3}{2}}}{-\partial_x^2 y}.$$

Hence, if the values of $\partial_x y$ and $\partial_x^2 y$ be obtained, in terms of x and y, or of one of them only, from the equation to the curve, and substituted in the above expressions for α, β, and R, the position and magnitude of the "circle of curvature" at any point (x, y) of the curve will be determined.

Cor. 1. The expression for R, the "radius of curvature", may be transformed into another, which is often more convenient of application, when $\partial_x y$ is of a fractional form:—

Thus $R = \dfrac{\left\{1 + (\partial_x y)^2\right\}^{\frac{3}{2}}}{-\partial_x^2 y}$.

Dividing numerator and denominator by $(\partial_x y)^3$,

$$R = \frac{\left\{1 + \dfrac{1}{(\partial_x y)^2}\right\}^{\frac{3}{2}}}{\frac{1}{2} . \partial_x \dfrac{1}{(\partial_x y)^2}};$$

or $R = \dfrac{2\left\{1 + (\partial_y x)^2\right\}^{\frac{3}{2}}}{\partial_x (\partial_y x)^2}$.

Cor. 2. If $u = f(x,y) = 0$ be the equation to the given curve, since (Art. 74.)

$$\partial_x y = -\frac{\partial_{(x)} u}{\partial_{(y)} u},$$

$$\text{and } \partial_x^2 y = -\frac{1}{(\partial_{(y)} u)^3}\{\partial_{(x)}^2 u.(\partial_{(y)} u)^2 - 2.\partial_{(x)}\partial_{(y)} u.\partial_{(x)} u.\partial_{(y)} u + \partial_{(y)}^2 u.(\partial_{(x)} u)^2\};$$

$$\therefore\ R = \frac{\{(\partial_{(x)} u)^2 + (\partial_{(y)} u)^2\}^{\frac{3}{2}}}{\partial_{(x)}^2 u.(\partial_{(y)} u)^2 - 2.\partial_{(x)}\partial_{(y)} u.\partial_{(x)} u.\partial_{(y)} u + \partial_{(y)}^2 u.(\partial_{(x)} u)^2}.$$

Obs. Since the circle has the same curvature at every point of it, and is of all curves the most easily described, it has been adopted as the measure of the curvature of all other curves; so that when the radius of the circle of curvature at any point of a proposed curve is given, the curvature of the curve at that point is sufficiently defined and understood.

Ex. Required the "radius of curvature" at any point of an ellipse.

Let $y = \frac{b}{a}\sqrt{a^2 - x^2}$ be the equation to the ellipse,

$$\text{then } \partial_x y = -\frac{b}{a}.\frac{x}{\sqrt{a^2 - x^2}},$$

$$\partial_x^2 y = -\frac{b}{a}.\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}} = -\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}};$$

$$\therefore\ R = \frac{\left(1 + \frac{b^2}{a^2}.\frac{x^2}{a^2 - x^2}\right)^{\frac{3}{2}}}{\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}}$$

$$= \frac{\left(\frac{a^2 y^2}{b^2} + \frac{b^2 x^2}{a^2}\right)^{\frac{3}{2}}}{ab}$$

$$= \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4},\ \text{for any point } (x,y).$$

If the point be the extremity of the axis major, $x = a$, $y = 0$,

$$\text{and } R = \frac{b^2}{a}.$$

If the point be the extremity of the axis minor, $x = 0$, $y = b$,

$$\text{and } R = \frac{a^2}{b}.$$

148. *To find an expression for the "Radius of Curvature", when the arc of the curve, measured from some fixed point in it, is the variable of differentiation.*

By the preceding Art., Cor. 1.,

$$R = \frac{\{1 + (\partial_y x)^2\}^{\frac{3}{2}}}{\frac{1}{2}.\partial_x(\partial_y x)^2}.$$

And if s be the arc, $(\partial_s x)^2 + (\partial_s y)^2 = (\partial_s s)^2 = 1$; (Art. 51).

$$\text{Hence } 1 + (\partial_y x)^2 = 1 + \left(\frac{\partial_s x}{\partial_s y}\right)^2,$$

$$= \frac{(\partial_s x)^2 + (\partial_s y)^2}{(\partial_s y)^2}$$

$$= \frac{1}{(\partial_s y)^2}.$$

$$\text{Also } \partial_x(\partial_y x)^2 = \frac{1}{\partial_s x}.\partial_s \frac{1}{(\partial_s y)^2}$$

$$= -\frac{1}{\partial_s x}.\frac{2.\partial_s^2 y}{(\partial_s y)^3};$$

$$\therefore\ R = -\frac{1}{(\partial_s y)^3}.\frac{\partial_s x.(\partial_s y)^3}{\partial_s^2 y}$$

$$= -\frac{\partial_s x}{\partial_s^2 y}.$$

Or similarly it may be shewn that

$$R = -\frac{\partial_s y}{\partial_s^2 x}.$$

Cor. Since $R^2.(\partial_s^2 x)^2 = (\partial_s y)^2$,

and $R^2.(\partial_s^2 y)^2 = (\partial_s x)^2$;

$$\therefore\ R^2.\{(\partial_s^2 x)^2 + (\partial_s^2 y)^2\} = (\partial_s x)^2 + (\partial_s y)^2 = 1;$$

$$\therefore\ R = \frac{1}{\sqrt{(\partial_s^2 x)^2 + (\partial_s^2 y)^2}}.$$

149. *To prove that the "Radius of Curvature" is a maximum or minimum when the contact of the circle of curvature with the curve is of the third order.*

$$R = \frac{\{1 + (\partial_x^2 y)^2\}^{\frac{3}{2}}}{-\partial_x^2 y};$$

$$\therefore\ \log_e R = \frac{3}{2}.\log_e \overline{1 + (\partial_x y)^2} - \log_e \overline{-\partial_x^2 y};$$

$$\therefore\ \partial_x \log_e R = \frac{3.\partial_x y.\partial_x^2 y}{1 + (\partial_x y)^2} - \frac{\partial_x^3 y}{\partial_x^2 y}.$$

But if the contact be of the third order, x, y, $\partial_x y$, $\partial_x^2 y$, and $\partial_x^3 y$ are the same in the curve as in the circle;

$$\therefore\ (x-\alpha)^2 + (y-\beta)^2 = R^2 \ldots\ldots\ldots(1),$$

$$x - \alpha + (y-\beta).\partial_x y = 0 \ldots\ldots\ldots(2),$$

$$1 + (\partial_x y)^2 + (y-\beta).\partial_x^2 y = 0 \ldots\ldots\ldots(3),$$

$$2.\partial_x y.\partial_x^2 y + \partial_x y.\partial_x^2 y + (y-\beta).\partial_x^3 y = 0 \ldots\ldots\ldots(4).$$

And from (4) $3.\partial_x y.\partial_x^2 y = -(y-\beta).\partial_x^3 y$,

... . (3) $1 + (\partial_x y)^2 = -(y-\beta).\partial_x^2 y$;

$$\therefore\ \frac{3.\partial_x y.\partial_x^2 y}{1 + (\partial_x y)^2} = \frac{\partial_x^3 y}{\partial_x^2 y}.$$

Hence $\partial_x \log_e R = 0$;

∴ (Art. 118.) R is a maximum or minimum.

150. *To find the locus of the centre of the circle of curvature to any curve.*

Since the co-ordinates of the centre of the circle of curvature corresponding to any point of a curve are functions of the co-ordinates of that point determined by the equations

$$\left.\begin{aligned} \alpha &= x - \frac{\partial_x y}{\partial_x^2 y}\{1 + (\partial_x y)^2\} \\ \beta &= y + \frac{1}{\partial_x^2 y}\{1 + (\partial_x y)^2\}; \end{aligned}\right\} \ldots\ldots\ldots\ldots (A)$$

if between these two equations and the given equation to the curve x and y be eliminated, the resulting equation between α and β will hold for all values of x and y, and is therefore the equation to the locus required.

The curve thus determined is called the Evolute, and the given curve is called the Involute to it.

Ex. To find the equation to the evolute of the ellipse.

$$\text{Here } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

$$\therefore \frac{x}{a^2} + y.\frac{\partial_x y}{b^2} = 0, \quad \text{or } \partial_x y = -\frac{b^2 x}{a^2 y}.$$

$$\text{Also } \frac{1}{a^2} + \frac{(\partial_x y)^2}{b^2} + y.\frac{\partial_x^2 y}{b^2} = 0;$$

$$\therefore \partial_x^2 y = -\frac{b^2}{a^2 y} - \frac{b^4 x^2}{a^4 y^3}$$

$$= -\frac{b^4}{a^2 y^3}\left\{\frac{y^2}{b^2} + \frac{x^2}{a^2}\right\}$$

$$= -\frac{b^4}{a^2 y^3}.$$

$$\text{And } \beta = y + \frac{1 + (\partial_x y)^2}{\partial_x^2 y};$$

$$\therefore \beta.\partial_x^2 y = y.\partial_x^2 y + 1 + (\partial_x y)^2,$$

or by substitution $-\beta.\dfrac{b^4}{a^2y^3} = -\dfrac{b^2}{a^2} + 1$

$$= \frac{a^2 - b^2}{a^2};$$

$$\therefore\ b\beta = -\frac{y^3}{b^3}(a^2 - b^2),$$

$$\text{or}\ (b\beta)^{\frac{2}{3}} = \frac{y^2}{b^2}(a^2 - b^2)^{\frac{2}{3}}.$$

Similarly it may be shewn that

$$(a\alpha)^{\frac{2}{3}} = \frac{x^2}{a^2}(a^2 - b^2)^{\frac{2}{3}};$$

$$\therefore\ (a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right).(a^2 - b^2)^{\frac{2}{3}},$$

$$\text{or}\quad (a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

151. *The normal to the curve is a tangent to the evolute.*

Since α and β are functions of x, differentiating with regard to x equations (A) in Art. 150, and representing $\dfrac{1 + (\partial_x y)^2}{\partial_x^2 y}$ by p,

$$\partial_x \alpha = 1 - \partial_x^2 y.p - \partial_x y.\partial_x p$$

$$= -(\partial_x y)^2 - \partial_x y.\partial_x p.$$

And $\partial_x \beta = \partial_x y + \partial_x p$;

$$\therefore\ \partial_x \alpha + \partial_x \beta.\partial_x y = 0,$$

$$\text{or}\ -\frac{1}{\partial_x y} = \frac{\partial_x \beta}{\partial_x \alpha}$$

$$= \partial_\alpha \beta.$$

Now the equation to the normal at the point (x,y), since it passes through the point (α,β), is

$$y-\beta=-\frac{1}{\partial_x y}.(x-\alpha);$$

$$\text{and } \therefore \text{ is } y-\beta=\partial_\alpha\beta.(x-\alpha),$$

which is the equation to the tangent to the evolute at the point (α,β) passing through the point (x,y). Hence the normal to the curve and the corresponding tangent to the evolute coincide.

152. *The arc of the evolute measured from a fixed point is always equal to the radius of curvature corresponding to the other extremity increased or diminished by a fixed quantity.*

From Art. 150,

$$\text{if } p \text{ represent } \frac{1+\partial_x y)^2}{\partial_x^2 y};$$

$$\alpha=x-\partial_x y.p,$$

$$\beta=y+p,$$

$$\text{and } R=\sqrt{1+(\partial_x y)^2}.p.$$

$$\therefore\ \partial_x\alpha=1-\partial_x^2 y.p-\partial_x y.\partial_x p$$

$$=-(\partial_x y)^2-\partial_x y.\partial_x p;$$

$$\text{and } \partial_x\beta=\partial_x y+\partial_x p;$$

$$\therefore\ (\partial_x\alpha)^2+(\partial_x\beta)^2=(\partial_x y+\partial_x p)^2\{1+(\partial_x y)^2\}.$$

But if S be the arc of the evolute $(\partial_x S)^2=(\partial_x\alpha)^2+(\partial_x\beta)^2$;

$$\therefore\ \partial_x S=(\partial_x y+\partial_x p).\sqrt{1+(\partial_x y)^2}.$$

$$\text{Also}\quad \partial_x R = \sqrt{1 + (\partial_x y)^2}.\partial_x p + p.\frac{\partial_x y.\partial_x^2 y}{\sqrt{1 + (\partial_x y)^2}}$$

$$= \sqrt{1 + (\partial_x y)^2}.\partial_x p + \partial_x y.\sqrt{1 + (\partial_x y)^2}$$

$$= (\partial_x y + \partial_x p).\sqrt{1 + (\partial_x y)^2};$$

$$\therefore\ \partial_x S = \partial_x R.$$

Hence $S = R \pm$ constant quantity.

Note. From the two preceding propositions it follows that if a string of sufficient length kept constantly stretched be unwound from the evolute of a curve, either the extremity or some point of the string will trace out the curve itself. Hence the term "evolute."

ASYMPTOTES.

Def. An asymptote to a curve is a line which, as the abscissa is continuously increased, approaches continuously to the curve, and becomes nearer to it than by any assignable quantity, but never meets it.

153. *To draw a rectilinear asymptote, if there be any, to a given curve referred to rectangular co-ordinates.*

If the curve admit of a rectilinear asymptote, it is evident that the tangent to the curve corresponding to an infinite abscissa may be considered as coincident with the asymptote.

Let PT (fig. Art. 125.), the tangent to the curve at P, be produced to meet the axis of y in t;

$AM(x)$, $MP(y)$ the co-ordinates of P.

Then $\quad AT = AM - TM$

$\quad = x - y.\partial_y x.$ (Art. 126).

Also $$\begin{aligned} At &= AT.\tan PTM \\ &= (x - y.\partial_y x).\partial_x y \\ &= x.\partial_x y - y\,; \end{aligned}$$

or, if At is supposed to be measured in a *positive* direction from A,

$$At = y - x.\partial_x y.$$

And, if these values of AT and At, when $x = \infty$, are both finite, the corresponding line Tt, indefinitely produced both ways, is a rectilinear asymptote to the curve.

If $AT_{x=\infty} = \infty$, and $At_{x=\infty}$ is finite, the line drawn at the distance $At_{x=\infty}$ parallel to AX is the asymptote.

If $At_{x=\infty} = \infty$, and $AT_{x=\infty}$ is finite, the line drawn at the distance $AT_{x=\infty}$ parallel to AY is the asymptote.

If $AT_{x=\infty} = \infty$, and also $At_{x=\infty} = \infty$, no rectilinear asymptote to the curve can be drawn.

If $AT_{x=\infty} = 0$, and $At_{x=\infty}$ is finite or ∞, AY, the axis of y, is the asymptote.

If $At_{x=\infty} = 0$, and $AT_{x=\infty}$ is finite or ∞, AX, the axis of x, is the asymptote.

If $AT_{x=\infty} = 0 = At_{x=\infty}$, the asymptote passes through A, and its direction will be determined from the value of $(\tan^{-1} \partial_x y)_{x=\infty}$.

Ex. To determine the rectilinear asymptotes to a Conic Section whose equation is $y^2 = mx + nx^2$.

$$2y.\partial_x y = m + 2nx\,;$$

$$\therefore\ \partial_y x = \pm \frac{2(mx + nx^2)^{\frac{1}{2}}}{m + 2nx}\,;$$

$$\begin{aligned} \therefore\ AT &= x - y.\partial_y x \\ &= x - \frac{2(mx + nx^2)}{m + 2nx} \\ &= -\frac{mx}{m + 2nx}. \end{aligned}$$

Also $At = y - x.\partial_x y$

$$= y \mp \frac{mx + 2nx^2}{2(mx + nx^2)^{\frac{1}{2}}}$$

$$= \pm \frac{mx}{2(mx + nx)^{\frac{1}{2}}}.$$

Or $AT = -\dfrac{m}{\dfrac{m}{x} + 2n}$,

and $At = \pm \dfrac{m}{2\sqrt{\dfrac{m}{x} + n}}$;

$$\therefore\ AT_{x=\infty} = -\frac{m}{2n}, \text{ and } At_{x=\infty} = \pm\frac{m}{2\sqrt{n}},$$

which values are finite if n be not either 0 or negative, i.e., if the Conic Section be neither a parabola, nor an ellipse. The hyperbola, therefore, has two asymptotes, corresponding to the two pairs of values of $AT_{x=\infty}$ and $At_{x=\infty}$.

154. *To find generally the curvilinear or rectilinear asymptotes, if there be any, to a plane curve.*

Let the equation to the curve be put, if possible, under the form

$$y = ax^m + bx^{m-1} + \ldots\ldots + px + q + \frac{r}{x} + \frac{s}{x^2} + \&c.$$

m being a positive integer.

Then, if $Y = ax^m + bx^{m-1} + \ldots\ldots + px + q$ be the equation to another curve referred to the same axes of co-ordinates, for the same abscissa (x)

$$y - Y = \frac{r}{x} + \frac{s}{x^2} + \&c.;$$

therefore, as x is continuously increased, the distance between the curves, measured by the difference of their ordinates, is continuously diminished, and becomes less than any assignable quantity, but never becomes equal to 0. Hence the curve whose equation is

$$Y = ax^m + bx^{m-1} + \ldots\ldots + px + q$$

approaches continuously to the given curve, and becomes nearer to it than by any assignable quantity, but never meets it, i. e., it is an "asymptote" to the given curve.

If $m = 1$, then the asymptote is a straight line, whose equation is $y = ax + b$.

If $m = 2$, the asymptote is the common parabola.

If m be any whole number greater than 2, the asymptote is what is called a parabolic curve of the m^{th} order.

As many series as can be found for y descending according to negative and integral powers of x, so many rectilinear or parabolic asymptotes does the curve admit of.

NOTE. Although it is usual to have regard to such asymptotes only as are mentioned above, it is evident that, according to the Definition, each of the curves represented by the equations

$$Y = ax^m + bx^{m-1} + \ldots + px + q + \frac{r}{x},$$

$$Y = ax^m + bx^{m-1} + \ldots + px + q + \frac{r}{x} + \frac{s}{x^2},$$

&c. = &c.

is an "asymptote" to the given curve.

COR. If $m = 1$, or the curve admit of a rectilinear asymptote, the series for y may be determined from the series proved in Art. 106.

For, by hypothesis, $\frac{y}{x}$ is capable of expansion in a series of descending integral powers of x, the index of the first power being 0. Therefore, if $\frac{y}{x} = fx$ be the equation to the curve,

$$\frac{y}{x} = \left(f\frac{1}{x}\right)_{x=0} + \left(\partial_x f\frac{1}{x}\right)_{x=0} \cdot \frac{x^{-1}}{\underline{|1}} + \left(\partial_x^2 f\frac{1}{x}\right)_{x=0} \cdot \frac{x^{-2}}{\underline{|2}} + \&c$$

(Art. 106.);

$$\therefore\ y = \left(f\frac{1}{x}\right)_{x=0} .x + \left(\partial_x f\frac{1}{x}\right)_{x=0} + \left(\partial_x^2 f\frac{1}{x}\right)_{x=0} \cdot \frac{1}{2x} + \&c.$$

$$\text{and} \therefore\ Y = \left(f\frac{1}{x}\right)_{x=0} .x + \left(\partial_x f\frac{1}{x}\right)_{x=0}$$

is the equation to the rectilinear asymptote.

Ex. 1. Required the parabolic and rectilinear asymptotes to the curve whose equation is $x^2y - 2ay^2 - a^2x = 0$.

$$x^2y - 2ay^2 - a^2x = 0,$$

$$\text{or}\ y^2 - \frac{x^2}{2a}.y + \left(\frac{x^2}{4a}\right)^2 = \left(\frac{x^2}{4a}\right) - \frac{ax}{2}$$

$$= \left(\frac{x^2}{4a}\right)^2 .\left\{1 - \frac{8a^3}{x^3}\right\};$$

$$\therefore\ y = \frac{x^2}{4a} \pm \frac{x^2}{4a}.\sqrt{1 - \frac{8a^3}{x^3}}$$

$$= \frac{x^2}{4a} \pm \frac{x^2}{4a}\left\{1 - \frac{4a^3}{x^3} - \frac{8a^6}{x^6} + \ldots\ldots\right\},$$

$$\therefore\ y = \frac{x^2}{2a} + 0.x + 0 - \frac{a^2}{x} - \ldots\ldots\ldots\ldots$$

$$\text{Also}\ y = 0.x + 0 + \frac{a^2}{x} + \ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, $Y = \frac{x^2}{2a}$ is the equation to a parabolic asymptote.

and $Y = 0.x$ .... rectilinear ....

Or the curve has a parabolic asymptote, which is a common parabola, with its vertex at the origin, its principal diameter the axis of y, and its latus rectum $= 2a$; and also a rectilinear asymptote coincident with the axis of x.

Ex. 2. Required the rectilinear asymptote to the curve whose equation is $x^3 + y^3 - 3axy = 0$.

Let $\frac{y}{x} = fx$, or $y = x.fx$, then

$$x^3 + x^3.(fx)^3 - 3ax^2.fx = 0;$$

$$\therefore\ 1 + (fx)^3 - \frac{3a}{x}.fx = 0.$$

and if $\frac{1}{z} = x$, $1 + \left(f\frac{1}{z}\right)^3 - 3az.f\frac{1}{z} = 0;$

$$\therefore\ 3.\left(f\frac{1}{z}\right)^2.\partial_z f\frac{1}{z} - 3a.f\frac{1}{z} - 3az.\partial_z f\frac{1}{z} = 0;$$

$$\therefore\ \left(f\frac{1}{x}\right)_{x=0} = \left(f\frac{1}{z}\right)_{z=0} = -1,$$

$$\text{and}\ \left(\partial_x f\frac{1}{x}\right)_{x=0} = \left(\partial_z f\frac{1}{z}\right)_{z=0} = -a;$$

$\therefore$ (Cor. Art. 154) $Y = -x - a$ is the equation to the asymptote.

III. CURVES REFERRED TO POLAR CO-ORDINATES.

155. *To determine the direction of curvature of a given plane curve at any point, the curve being referred to polar co-ordinates.*

The direction of curvature of a curve at any point is the same with the direction of its tangent at that point. Let P be the point, (Vide Fig. Art. 53) whose co-ordinates are $\angle ASP(\theta)$, $SP(r)$; then PT being the tangent to the curve at P,

$$\text{(Art. 53)}\quad \tan SPT = r.\partial_r\theta,$$

$$\therefore\ \angle SPT = \tan^{-1}(r.\partial_r\theta).$$

Hence, if the value of $\partial_r\theta$ in terms of r and θ, or of one of them only, be obtained from the given equation to the curve, the value of the angle SPT, or the direction of the curve at any point, may be found by substituting in the above equation.

Obs. Of the two angles which SP makes with the tangent at P, that which is equal to $\tan^{-1} r.\partial_r\theta$ is the one which is on the same side of SP as the angle θ.

156. *To find the "subtangent" for any point of a plane curve referred to polar co-ordinates.*

Def. The line drawn from the pole at right angles to the radius vector of any point of the curve and terminated by the tangent at that point is called the "subtangent".

Let $\angle ASP(\theta)$, $SP(r)$ be the co-ordinates of the point P;

PT the tangent at P; and let SZ be drawn at right angles to SP and meeting the tangent in Z (Fig. Art. 157); then SZ is the "subtangent" for the point P.

$$\begin{aligned} \text{And } SZ &= SP.\tan SPT \\ &= r.r.\partial_r\theta, \quad \text{(Art. 53)} \\ &= r^2.\partial_r\theta. \end{aligned}$$

Hence, if the value of $\partial_r\theta$ in terms of r and θ, or of one of them only, be obtained from the given equation to the curve, and substituted in the above expression for SZ, the "subtangent" will be determined for any point $(r, \theta,)$ of the curve, in terms of r and θ, or of one of them only.

Obs. Since $\partial_\theta \frac{1}{r} = -\frac{\partial_\theta r}{r^2} = -\frac{1}{SZ}$, if, in the given equation to the curve, r be an explicit function of θ, the process of finding the subtangent may frequently be facilitated by differentiating $\frac{1}{r}$ instead of r. Thus,

Ex. Required the subtangent for any point of an hyperbola.

Here $$r = \frac{a\,(e^2-1)}{1+e.\cos\theta},$$

$$\therefore\ \frac{1}{r} = \frac{1+e.\cos\theta}{a\,(e^2-1)};$$

$$\therefore\ \frac{\partial_\theta r}{r^2} = \frac{e.\sin\theta}{a\,(e^2-1)},$$

and $\frac{r^2}{\partial_\theta r} = \frac{a\,(e^2-1)}{e.\sin\theta}$ = subtangent for any point (r,θ).

157. *To determine whether a plane curve referred to polar co-ordinates is "concave" or "convex" to the pole at any point.*

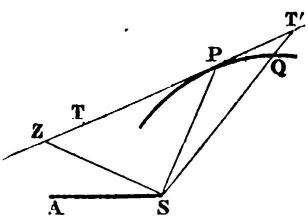
Def. A curve is said to be concave or convex to the pole at any point according as the radii vectores to the immediately preceding and succeeding points are *both* less or *both* greater than the corresponding radii to the tangent at that point.

Let $SP(r)$, $ASP(\theta)$ be the polar co-ordinates of the point P;
SQ, $ASQ(\theta+h)$ .... Q.

Produce SQ, if necessary, to meet the tangent at P in T', and let the angle $SPT = \phi$. Then the curve will be concave or convex to the pole according as $ST' - SQ$, when h is indefinitely diminished, is positive or negative independently of the sign of h.

But $$ST' = SP.\frac{\sin SPT'}{\sin ST'P}$$

$$= r.\frac{\sin\phi}{\sin\overline{\phi-h}}.$$



And since $$\partial_\phi\frac{1}{\sin\phi} = -\frac{\cos\phi}{(\sin\phi)^2},$$

$$\partial^2_\phi\frac{1}{\sin\phi} = \frac{1+(\cos\phi)^2}{(\sin\phi^3)},$$

&c. = &c.

$$\frac{1}{\sin\overline{\phi - h}} = \frac{1}{\sin\phi} + \frac{\cos\phi}{(\sin\phi)^2}.h + \frac{1+(\cos\phi)^2}{(\sin\phi)^3}.\frac{h^2}{\underline{|2}} + \ldots\ldots \text{(Art. 100.)}$$

$$\therefore\ ST' = r + r.\frac{\cos\phi}{\sin\phi}.h + r.\frac{1+(\cos\phi)^2}{(\sin\phi)^2}.\frac{h^2}{\underline{|2}} + \ldots\ldots$$

$$= r + r.\cot\phi.h + r.\{1 + 2(\cot\phi)^2\}.\frac{h^2}{\underline{|2}} + \ldots\ldots$$

And since $\cot\phi = \frac{\partial_\theta r}{r}$, (Art. 53.)

$$ST' = r + \partial_\theta r.h + r.\left\{1 + 2.\left(\frac{\partial_\theta r}{r}\right)^2\right\}.\frac{h^2}{\underline{|2}} + \ldots\ldots$$

Also, since SQ is the same function of $\overline{\theta + h}$ that SP is of θ,

$$SQ = r + \partial_\theta r.h + \partial_\theta^2 r.\frac{h^2}{\underline{|2}} + \ldots\ldots$$

$$\therefore\ ST' - SQ = \{r + \frac{2}{r}.(\partial_\theta r)^2 - \partial_\theta^2 r\}.\frac{h^2}{\underline{|2}} + \ldots\ldots$$

But, by sufficiently diminishing h, the sign of the latter series may be made the same with the sign of its first term (Art. 100. Cor.) which sign will not change with the sign of h. Hence, when h is indefinitely diminished, $ST' - SQ$ will have the same sign independently of the sign of h, and the curve will be "concave" or "convex" at the point (r, θ) according as the sign of the coefficient of the first term is positive or negative, i. e. according as

$$\partial_\theta^2 r - \frac{2}{r}.(\partial_\theta r)^2 \text{ is } < \text{ or } > r.$$

Cor. If $\partial_\theta^2 r - \frac{2}{r}.(\partial_\theta r)^2 = r$, and the coefficient of h^3 in the series for $ST' - SQ$ be not also equal to 0, the point P is a point of "contra-flexure".

Obs. Since $\partial_\theta^2 \dfrac{1}{r} = -\dfrac{\partial_\theta^2 r}{r^2} + \dfrac{2(\partial_\theta r)^2}{r^3} = -\dfrac{1}{r^2}\left\{\partial_\theta^2 r - \dfrac{2}{r}.(\partial_\theta r)^2\right\}$, it is evident that the process for determining whether a curve is "concave" or "convex" at any point may frequently be abridged by successively differentiating $\dfrac{1}{r}$ instead of r; and, if $\dfrac{1}{r} = u$, then the curve will be "concave" or "convex" at the point (r,θ), according as

$$-\partial_\theta^2 u \text{ is} < \text{or} > u.$$

Also, there may be a point of "contra-flexure" if $-\partial_\theta^2 u = u$.

Ex. Required to determine whether the curve, whose equation is $r^2 = \dfrac{a^2}{\theta}$, is "concave" or "convex" at any point.

$$\text{Here } au = \sqrt{\theta};$$

$$\therefore\ a.\partial_\theta u = \frac{1}{2\sqrt{\theta}},$$

$$\text{and } a.\partial_\theta^2 u = -\frac{1}{4\theta^{\frac{3}{2}}};$$

$$\therefore\ -\partial_\theta^2 u = \frac{1}{4a^4u^3},$$

which is $<$ or $> u$, according as $4a^4u^4$ is $>$ or < 1; i.e. according as a^2u^2 is $>$ or $< \frac{1}{2}$; or $\theta >$ or $< \frac{1}{2}$. Hence, the curve is "concave" for all values of θ from $\frac{1}{2}$ to ∞, and "convex" for all values of θ from 0 to $\frac{1}{2}$.

If $-\partial_\theta^2 u = u$, or $\theta = \frac{1}{2}$, the corresponding point of the curve is a point of "contra-flexure".

158. Let two plane curves be referred to polar coordinates having the same "pole" and "prime radius"; and let ρ, r be the radii vectores for a common angle θ, and $\rho = \phi\theta$, $r = f\theta$ the equations to the curves; then

$$\phi\overline{\theta+h}-f\overline{\theta+h}=\rho-r+(\partial_\theta\rho-\partial_\theta r).h+(\partial_\theta^2\rho-\partial_\theta^2 r).\frac{h^2}{\underline{|2}}+\&c.$$

If now $\rho=r$, the two curves meet at the point (r,θ).
If also $\partial_\theta\rho=\partial_\theta r$, ... have a contact of the 1st order.
...... $\partial_\theta^2\rho=\partial_\theta^2 r$, ... have a contact of the 2nd order.

And generally, if such equations hold as far as $\partial_\theta^n\rho=\partial_\theta^n r$, the two curves have a contact which is said to be of the nth order at the point corresponding to angle θ.

It is obvious that the above series expresses the distance between the curves, measured along the radius vector corresponding to a common angle $\overline{\theta+h}$, which, when h is indefinitely diminished, is less or greater, according as the order of contact is higher or lower.

159. Since x, y, the rectangular co-ordinates of any point in a curve, considered as referred to *any* axes of rectangular co-ordinates, are determinate functions of the corresponding polar co-ordinates, r and θ, whatever may be the equation to the curve, it is evident that $\partial_x y$ is also a determinate function of r, θ and $\partial_\theta r$. Hence if r, θ and $\partial_\theta r$ be the same in any two curves referred to polar co-ordinates, the corresponding values of x, y, and $\partial_x y$ will be the same in the two curves, if referred to rectangular co-ordinates:—or, curves which have a contact of the 1st order at any point considered as polar curves have also a contact of the 1st order at the same point considered as referred to rectangular co-ordinates, and *vice versâ*. And the same may be shewn of any higher order of contact.

160. As in Art. 145, the polar equation to a straight line being of the form $\rho.\cos\theta+a\rho.\sin\theta+b=0$, (Hymer's Conic Sections, Art. 36.) in which there are *two* arbitrary constants, a straight line may be made to have a contact of the 1st order with a given curve at any point; and the equation to a circle being of the form

$\rho^2-2(a.\cos\theta+b.\sin\theta)\rho+a^2+b^2-c^2=0$, (Conic Sections, 51.) which contains *three* arbitrary constants, the circle may be made to have a contact of the 2nd order with a curve at any point:—as would appear also from Art. 159, connected with Art. 145.

161. *To determine the circle of curvature at any proposed point of a given curve referred to polar co-ordinates.*

Let a and β be the polar co-ordinates of the centre, and R the radius, of the circle of curvature at the point (r, θ) of the curve.

Then, for the point of contact, the three following equations

$$r^2 - 2a.\cos\overline{\theta-\beta}.r + a^2 - R^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$r.\partial_\theta r - a.\cos\overline{\theta-\beta}.\partial_\theta r + a.\sin\overline{\theta-\beta}.r = 0 \ldots\ldots\ldots\ldots (2)$$

$$r.\partial_\theta^2 r + (\partial_\theta r)^2 - a.\cos\overline{\theta-\beta}.\partial_\theta^2 r + 2a.\sin\overline{\theta-\beta}.\partial_\theta r + a.\cos\overline{\theta-\beta}.r = 0 \ldots (3),$$

hold for all corresponding values of r, θ, $\partial_\theta r$, and $\partial_\theta^2 r$.

Hence from these three equations a, β, and R may be determined in terms of r, θ, $\partial_\theta r$, and $\partial_\theta^2 r$.

It will appear by elimination that

$$a = \pm\frac{r\sqrt{r^2.\{r^2+(\partial_\theta r)^2\} + (\partial_\theta r)^2.\{r^2 + 3(\partial_\theta r)^2 - 2r.\partial_\theta^2 r\}}}{r^2 + 2(\partial_\theta r)^2 - r.\partial_\theta^2 r} \ldots\ldots (\text{i})$$

$$\beta = \theta + \tan^{-1}\frac{\partial_\theta r\{r^2 + (\partial_\theta r)^2\}}{r\{r.\partial_\theta^2 r - (\partial_\theta r)^2\}} \ldots\ldots\ldots\ldots\ldots\ldots (\text{ii})$$

$$R = \frac{\{r^2 + (\partial_\theta r)^2\}^{\frac{3}{2}}}{r^2 + 2(\partial_\theta r)^2 - r.\partial_\theta^2 r} \ldots\ldots\ldots\ldots\ldots\ldots (\text{iii}).$$

Cor. If $r = \frac{1}{u}$, then $\partial_\theta r = -\frac{\partial_\theta u}{u^2}$, and

$$\partial_\theta^2 u = u^3.\{2.(\partial_\theta r)^2 - r.\partial_\theta^2 r\};$$

$$\therefore\ R = \frac{1}{u^3}.\frac{\{u^2 + (\partial_\theta u)^2\}^{\frac{3}{2}}}{u + \partial_\theta^2 u}.$$

Ex. Required to determine the circle of curvature at any point of the curve whose equation is $r = \epsilon^\theta$.

Here $\partial_\theta r = \partial_\theta^2 r = r$; therefore substituting in equations (1) (2) and (3), and eliminating

$$a = \pm r$$

$$\beta = \theta + 90^0$$

$$R = \sqrt{2}.r.$$

Cor. From equations (i) and (ii) and the given equation to the curve if r and θ be eliminated, the resulting equation between a and β will be the polar equation to the Evolute.

162. The equations to curves are sometimes given between r, the radius vector, and p, the perpendicular from the pole on the tangent; and although such equations will not serve to determine the form of the curve, many of the properties of the curve may be determined from them. In fact, since in all cases $p = \dfrac{r^2}{\sqrt{r^2 + (\partial_\theta r)^2}}$, the equation to a curve, given between r and p, is equivalent only to that which would be obtained by differentiating its equation between r and θ.

163. *To find an expression for the Radius of Curvature of a curve when its equation is given in terms of* r *and* p.

From Art. 54. Cor. 3. $p = \dfrac{1}{\sqrt{u^2 + (\partial_\theta u)^2}}$, if $r = \dfrac{1}{u}$,

$$\therefore \partial_\theta p = -\frac{u.\partial_\theta u + \partial_\theta u.\partial_\theta^2 u}{\{u^2 + (\partial_\theta u)^2\}^{\frac{3}{2}}}$$

$$= -\partial_\theta u.\frac{u + \partial_\theta^2 u}{\{u^2 + (\partial_\theta u)^2\}^{\frac{3}{2}}}$$

$$= -\partial_\theta u.\frac{1}{Ru^3}, \quad \text{(Art. 161., Cor.)};$$

$$\therefore R = -\frac{\partial_\theta u}{u^3}.\frac{1}{\partial_\theta p}$$

$$= r.\frac{\partial_\theta r}{\partial_\theta p}$$

$$= r.\partial_p r.$$

Cor. By similar triangles it is evident that the Chord of the circle of curvature drawn through the "pole" is equal to $2p.\partial_p r$.

164. The result obtained in the last Art. may be more directly arrived at in the following manner:

Let r be the radius vector to any point of the given curve,

p the corresponding perpendicular on the tangent;

d the distance from the pole of the centre of a circle which passes through the point (r, p), and has the same tangent as the curve;

R' the radius of the circle; ρ and π the co-ordinates to any point in it.

Then $d^2 = R^2 + \rho^2 - 2\pi R$ is the equation to the circle;

and $$0 = 2\rho.\partial_\pi \rho - 2R;$$

$$\therefore\ R = \rho.\partial_\pi \rho;$$

which gives the radius of the circle in terms of ρ and $\partial_\pi \rho$.

Now in any curve $p = \dfrac{r^2}{\sqrt{r^2 + (\partial_\theta r)^2}}$; hence $\partial_p r$ will not contain differentials of θ with regard to r of a higher order than the second: and therefore, if the above circle be the "circle of curvature", $\partial_p r = \partial_\pi \rho$; also $r = \rho$ at the point of contact. Hence $r.\partial_p r = \rho.\partial_\pi \rho$; and if R be the "Radius of Curvature" for the point (r, p) of the curve,

$$R = r.\partial_p r.$$

Ex. Required the "Radius of Curvature", and chord of curvature through the focus, for any point of an ellipse.

The equation to an ellipse in terms of r and p is

$$p^2 = \frac{b^2 r}{2a - r},$$ (Hymers' *Conic Sections*, Art. 132.);

$$\therefore\ 2\log_e p = \log_e b^2 + \log_e r - \log\overline{2a - r},$$

$$\text{and} \qquad \frac{2}{p} = \frac{\partial_p r}{r} + \frac{\partial_p r}{2a - r}$$

$$= \frac{2a}{r.(2a - r)}.\partial_p r;$$

$$\therefore\ R = r.\partial_p r$$

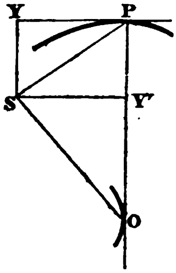
$$= \frac{r^2.(2a - r)}{ap}$$

$$= \frac{r^{\frac{3}{2}}.(2a - r)^{\frac{3}{2}}}{ab}.$$

Again, Chord of Curvature through the focus $= 2p.\partial_p r$

$$= \frac{2r.(2a - r)}{a}.$$

165. *To find the Evolute of a curve when the equation to the curve is given between* r *and* p.



Let PO be the Radius of curvature to the curve at P, and therefore O a point in the evolute. Then

$$PO = r.\partial_p r.$$

Also PO is a tangent to the evolute at the point O; therefore if SY' be drawn perpendicular to PO, SO and SY' are respectively the radius vector and perpendicular on the tangent to the evolute. Let $SO = \rho_1$, and $SY' = \pi_1$; then since $SY' = PY$, and $PY' = SY$,

$$\rho_1^2 - \pi_1^2 = OY'^2$$

$$= (r.\partial_p r - p)^2 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{and} \quad \pi_1^2 = r^2 - p^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

from which two equations, together with the given equation to the curve between r and p, by eliminating r and p an equation will be obtained between ρ and π, which is therefore the equation to the Evolute.

Ex. Required the evolute of the "logarithmic spiral" whose equation is $p = r.\sin\alpha$.

$$\text{Here } \partial_p r = \frac{1}{\sin\alpha},$$

$$\therefore\ \rho_1^2 - \pi_2^2 = \left(\frac{r}{\sin\alpha} - r.\sin\alpha\right)^2$$

$$= r^2.\frac{(\cos\alpha)^4}{(\sin\alpha)^2}.$$

$$\text{And } \pi_1^2 = r^2 - r^2(\sin\alpha)^2$$

$$= r^2(\cos\alpha)^2;$$

$$\therefore\ r^2 = \frac{\pi_1^2}{(\cos\alpha)^2},$$

$$\text{and } \rho_1^2 - \pi_1^2 = \pi_1^2.(\operatorname{cotan}\alpha)^2;$$

$$\therefore\ \pi_1^2.(\operatorname{cosec}\alpha)^2 = \rho_1^2,$$

$$\text{or } \pi_1 = \rho_1.\sin\alpha,$$

the equation to the Evolute; which is therefore similar and equal to its involute.

166. *To draw a rectilinear asymptote, if there be any, to a curve referred to polar co-ordinates.*

When a curve, which is given by its equation referred to polar co-ordinates r and θ, admits of a rectilinear asymptote, the asymptote may be drawn in the following manner:—Find the value of θ, when $r = \infty$, and draw the line SP which makes that angle with the "prime radius"; then find the value of the subtangent SZ, when $r = \infty$, which will evidently be finite, if there be an asymptote; draw this line at right angles to SP, and through Z draw a straight line parallel to SP; this last line will be the asymptote required.

If the value of $SZ_{r=\infty}$ should be negative, it must be drawn from S on the opposite side of SP to that on which the angle SPT is measured. (Vide Art. 155.)

Ex. To draw an asymptote to the hyperbola whose equation is $r = \frac{a(e^2-1)}{1+e.\cos\theta}$.

$$\text{If } r = \infty, \quad 1 + e.\cos\theta = 0;$$

$$\therefore \cos\theta = -\frac{1}{e},$$

$$\text{and } \theta = \cos^{-1} - \frac{1}{e}.$$

Also (Art. 156. Ex.)

$$SZ \text{ (the subtangent)} = \frac{r^2}{\partial_\theta r} = \frac{a(e^2-1)}{e.\sin\theta},$$

$$\therefore SZ_{r=\infty} = \frac{a(e^2-1)}{e\sqrt{1-\frac{1}{e^2}}}$$

$$= a.\sqrt{e^2-1}.$$

Hence the asymptote makes an angle $\cos^{-1} - \frac{1}{e}$ with the "prime radius", and its distance from the pole is $a.\sqrt{e^2-1}$.

IV. TRACING OF PLANE CURVES, &c.

167. *Having given the equation to a plane curve referred to rectangular co-ordinates, to find the form of the curve.*

By giving various values separately to x, and finding from the equation to the curve the corresponding values of y, it is evident that any number of points in the curve may be found, and its form be thus determined to any required degree of accuracy, provided the given equation is, or can be, solved with respect to y.

But the form of the curve may often be sufficiently determined by finding its nature at some principal points, or detecting some of its properties. For this purpose, find the values of $\partial_x y$ and $\partial_x^2 y$ from the given equation to the curve $f(x,y)=0$ in terms of x and y, or of one of them, as x, only. Then

$\{f(x,y)\}_{x=0}=0$ will give the points at which the curve meets the axis of y;

$\{f(x,y)\}_{y=0}=0$ will give the points at which the curve meets the axis of x;

$\partial_x y=0$ will give the points at which y may be a maximum or minimum;

$\partial_x^2 y=0$ will give the points at which there may be contra-flexure.

Also $\partial_x y$ will serve to determine the direction of curvature at any point; and the sign of $\partial_x^2 y$ to determine whether the curve is concave or convex to the axis of x at any point.

If $\partial_x y$ should be of the form $\frac{0}{0}$ for any values of x, the nature of the corresponding point may be determined by Art. 141.

The determination of the "asymptotes", if there be any, will also serve in some degree to fix the form of the curve, and they may be drawn by Arts. 153, 154.

The preceding method of determining the form of a curve will, however, in practice, sometimes be found troublesome from the difficulty of the operations. In such cases the following method may frequently be used with advantage:—

168. *To approximate to the form of a plane curve, given by its equation referred to rectangular co-ordinates, by finding parabolic curves, which coincide with the required curve at the origin for very small arcs, and "asymptotes" to the infinite branches.*

The co-ordinates being taken so that the curve passes through the origin, assume $y=Ax^a$; then to determine the parabolic curves which coincide with the given curve at the origin, substitute this value of y in the given equation, and assume a such that the index of x in two of the terms is the same and less than any other; when x is so small that higher powers of x may be neglected compared with lower, the equation will be reduced to those two terms; which will determine A and a. Hence at the origin the curve coincides with the parabolic curve whose equation is $y = Ax^a$.

If a should be equal to 1, the equation $y = Ax^a$ will determine only the direction of curvature at the origin; and whether the curve is concave or convex cannot be known from it. In this case assume $y = Ax + Bx^\beta$; then, substituting this value of y in the given equation, the quantity $y - Ax$ will disappear; and, assuming $\beta > 1$ and such that two of the indices may be the same and less than any other, when x is very small the equation will be reduced to those two terms; which will determine B and β. Hence at the origin the curve coincides with the parabolic curve whose equation is $y = Ax + Bx^\beta$.

If a be determined such that two of the indices may be the same and *greater* than any other, the equation will be reduced to the two corresponding terms when $x = \infty$. And the curve whose equation is $y = Ax^a$ will be an "asymptote" to the given curve. In this case, if $a = 1$, proceed as before, determining β such that two of the indices may be equal and *greater* than any other. The resulting equation $y = Ax + Bx^\beta$ will also be an "asymptote". If $\beta = 0$, the curve has a *rectilinear* asymptote whose equation is $y = Ax + B$.

Ex. 1. The equation to a curve is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, required its form at the origin of co-ordinates.

If $x = 0$, $y = 0$; therefore the curve passes through the origin. Assume $y = Ax^a$, then

$$(x^2 + A^2x^{2a})^2 = a^2(x^2 - A^2x^{2a}),$$

$$\text{or } x^4 + 2A^2x^{2a+2} + A^4x^{4a} = a^2x^2 - a^2A^2x^{2a};$$

Let $2\alpha = 2$, or $\alpha = 1$; then, neglecting the other terms,

$$a^2x^2 - a^2A^2x^3 = 0,$$

$$\text{or}\quad A^2 = 1,$$

$$\text{or}\quad A = \pm 1;$$

$$\therefore\ y = \pm x.$$

Hence the curve passes through the origin in two directions, making angles of 45° and 135° with the axis of x; and, to find the *parabolic* curves with which it coincides, assume $y = \pm x + Bx^\beta$, and proceed as before.

Ex. 2. The equation to a curve is $(ay^6 - x^7)^2 = a^4x^8y^2$, to approximate to its form.

If $x = 0$, $y = 0$; therefore the curve passes through the origin. Assume $y = Ax^\alpha$, then

$$(aA^6x^{6\alpha} - x^7)^2 = a^4A^2x^{2\alpha+8},$$

$$\text{or}\quad a^2A^{12}x^{12\alpha} - 2aA^6x^{6\alpha+7} + x^{14} = a^4A^2x^{2\alpha+8}.$$

Let $2\alpha + 8 = 14$, or $\alpha = 3$, then neglecting the other terms,

$$x^{14} = a^4A^2x^{14},$$

$$\text{or}\quad A = \pm\frac{1}{a^2};$$

$$\therefore\ y = \pm\frac{1}{a^2}x^3,$$

which determines two branches of the curve for very small arcs, each of which, as it may easily be shewn, *touches* the axis of x, and has a point of contraflexure at the origin.

Again, let $12\alpha = 2\alpha + 8$, or $\alpha = \frac{4}{5}$, then neglecting the other terms, which are higher powers of x,

$$a^2A^{12}x^{12\alpha} = a^4A^2x^{12\alpha},$$

$$\text{or}\quad A^{10} = a^2;$$

$$\therefore\ A = \pm a^{\frac{1}{5}},$$

$$\text{and}\quad y = \pm a^{\frac{1}{5}}x^{\frac{4}{5}},$$

which determines two more branches, each of which, as it may easily be shewn, touches the axis of y, and forms a "cusp of the 1st species", the two cusps lying on opposite sides of the origin.

Again, to find an asymptote to the infinite branches,

$$\text{let } 6a + 7 = 14, \quad \text{or } a = \frac{7}{6},$$

then neglecting the other term which is a lower power of x,

$$a^2 A^{12} x^{14} - 2aA^6 x^{14} + x^{14} = 0,$$

$$\text{or } aA^6 - 1 = 0;$$

$$\therefore\ A^6 = \frac{1}{a},$$

$$\text{and } y = \frac{1}{a^{\frac{1}{6}}} x^{\frac{7}{6}},$$

which gives an "asymptote" consisting of two similar and equal branches, each of which touches the axis of x at the origin and lies wholly on the positive side of it.

Hence, having determined the nature of the curve at the origin and an asymptote to its infinite branches, a considerable approximation to its general form has been attained.

169. *If a series of curves be described by the continuous variation of some constant which enters the general equation including all of them, to find the locus of the intersection of two consecutive ones.*

Let $f(x,y,a) = 0$ be the equation which represents the whole series of curves by the variation of a; and let a and $a+h$ be two values determining two corresponding curves whose equations are

$$f(x,y,a) = 0, \quad \text{and } f(x,y,\overline{a+h}) = 0;$$

$$\text{or } f(x,y,a) = 0,$$

$$\text{and } f(x,y,a) + \partial_{(a)} f(x,y,a).h + \partial^2_{(a)} f(x,y,a) \frac{h^2}{\underline{|2}} + \&c. = 0.$$

And if we suppose x,y the point in which these two curves intersect, for those values of x and y the above equations hold together; therefore for the same values

$$\partial_{(a)}f(x,y,a).h + \partial^2_{(a)}f(x,y,a).\frac{h^2}{2} + \&c. = 0,$$

$$\text{or } \partial_{(a)}f(x,y,a) + \partial^2_{(a)}f(x,y,a).\frac{h}{2} + \&c. = 0.$$

But by the hypothesis the two assumed curves must be consecutive, therefore $a+h$ must differ from a by a quantity less than any which can be assigned, that is, h must be less than any assignable quantity, and consequently the sum of the terms after the first in the above series will be less than any assignable quantity, or

$$\partial_{(a)}f(x,y,a) = 0.$$

Hence the elimination of a between the two equations

$$f(x,y,a) = 0, \quad \text{and } \partial_{(a)}f(x,y,a) = 0$$

will give the locus required.

170. *The curve determined in the last Art. touches all the curves of the series.*

For since a is a function of x and y determined by the equation $\partial_{(a)}f(x,y,a) = 0$, the equation for determining $\partial_x y$ is

$$\partial_{(x)}f(x,y,a)+\partial_{(y)}f(x,y,a).\partial_x y+\partial_{(a)}f(x,y,a).\partial_x a=0, \text{ (Art. 76.)}$$

which becomes

$$\partial_{(x)}f(x,y,a) + \partial_{(y)}f(x,y,a).\partial_x y = 0.$$

Also the equation for determining $\partial_x y$ in one of the curves corresponding to a given value of a is

$$\partial_{(x)}f(x,y,a) + \partial_{(y)}f(x,y,a).\partial_x y = 0;$$

therefore, since for the point in which the two curves meet the values of x and y are the same in each, and also the equations for determining $\partial_x y$ in terms of these quantities are the same, it follows that $\partial_x y$ is the same; and therefore the curves have a common tangent at that point, i. e. they touch each other.

Ex. To find the curve which bounds all the parabolas described in one vertical plane by a body projected from a given point with a given velocity in all possible directions in that plane.

Let the point from which the body is projected be the origin of co-ordinates, the axis of x being horizontal and of y vertical. Let h be the height due to the velocity of projection, and α the angle of projection for any one of the curves; then (Whewell's *Mechanics*),

the equation to any of the parabolas is

$$y = x.\tan\alpha - \frac{x^2}{4h}(\sec\alpha)^2.$$

Therefore, differentiating with respect to α,

$$0 = x.(\sec\alpha)^2 - \frac{x^2}{2h}.(\sec\alpha)^2.\tan\alpha\,;$$

$$\therefore\ 0 = 1 - \frac{x}{2h}.\tan\alpha, \quad\text{or}\quad \tan\alpha = \frac{2h}{x}.$$

$$\therefore\ y = 2h - \frac{x^2}{4h}\left(1 + \frac{4h^2}{x^2}\right)$$

$$= h - \frac{x^2}{4h},$$

$$\therefore\ x^2 = 4h.(h - y).$$

Hence the curve required is a parabola, whose focus is the point of projection, principal diameter the vertical line through that point, and latus rectum equal to four times the space due to the velocity of projection.

APPENDIX.

I. *To differentiate x^n directly from the Definition, (Art. 6.), when n is a positive integer, without assuming the Binomial Theorem.*

$$\Delta_x x^n \text{ is } (x+h)^n - x^n;$$

$$\therefore \frac{\Delta_x x^n}{h} = \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$= (x+h)^{n-1} \cdot \frac{1 - \left(\dfrac{x}{x+h}\right)^n}{1 - \dfrac{x}{x+h}},$$

by actual division

$$= (x+h)^{n-1} \cdot \left\{1 + \frac{x}{x+h} + \left(\frac{x}{x+h}\right)^2 + \ldots\ldots + \left(\frac{x}{x+h}\right)^{n-1}\right\};$$

$\therefore$ substituting 0 for h,

$$\partial_x x^n = x^{n-1} \cdot \{1 + 1 + 1 + \ldots \text{ to } n \text{ terms}\}$$

$$= n x^{n-1}.$$

II. *To differentiate $x^{\frac{m}{n}}$, where m and n are positive integers.*

$$\frac{\Delta_x x^{\frac{m}{n}}}{h} = \frac{(x+h)^{\frac{m}{n}} - x^{\frac{m}{n}}}{(x+h) - x}$$

$$= (x+h)^{\frac{m}{n}-1} \cdot \frac{1 - \left(\dfrac{x}{x+h}\right)^{\frac{m}{n}}}{1 - \dfrac{x}{x+h}};$$

$$\text{or, if } \left(\frac{x}{x+h}\right)^{\frac{1}{n}} = z = (x+h)^{\frac{m}{n}-1}\cdot\frac{1-z^m}{1-z^n};$$

dividing Numerator and Denominator of the fraction by $1-z$

$$= (x+h)^{\frac{m}{n}-1}\cdot\frac{1+z+z^2+\ldots+z^{m-1}}{1+z+z^2+\ldots+z^{n-1}};$$

∴ substituting 0 for h,

$$\partial_x x^{\frac{m}{n}} = x^{\frac{m}{n}-1}\cdot\frac{1+1+1+\ldots\text{to } m \text{ terms}}{1+1+1+\ldots\text{to } n \text{ terms}}$$

$$= \frac{m}{n}.x^{\frac{m}{n}-1}.$$

III. *To differentiate* x^{-n}, *and* $x^{-\frac{m}{n}}$.

$$\frac{\Delta_x x^{-n}}{h} = \frac{(x+h)^{-n}-x^{-n}}{(x+h)-x}$$

$$= -\frac{1}{x^n.(x+h)^n}.\frac{(x+h)^n-x^n}{(x+h)-x}$$

$$= -\frac{1}{x^n.(x+h)^n}.(x+h)^{n-1}.\left\{1+\frac{x}{x+h}+\ldots+\left(\frac{x}{x+h}\right)^{n-1}\right\}, \text{ as in I.}$$

∴ substituting 0 for h,

$$\partial_x x^{-n} = -\frac{x^{n-1}}{x^{2n}}.n$$

$$= -n.x^{-n-1}.$$

$$\text{Similarly } \partial_x x^{-\frac{m}{n}} = -\frac{m}{n}.x^{-\frac{m}{n}-1}.$$

IV. *To find the limits of the difference between the value of* $f\overline{x+h}$ *and the first* n *terms of its expansion by Taylor's Theorem, when a particular value* x_1 *is given to* x, *and* h *is a finite quantity.*

If $\frac{fz - fx}{z - x}$ be represented by $\phi(z, x)$, and n be substituted for $n - 1$ in equation $(N - 1)$ Art. 99,

$$0 = \partial_x^n fx + \partial_x^n \phi(z,x).(z - x) - n.\partial_x^{n-1}\phi(z,x)$$

$$= \partial_x^n fx.(z - x)^{n-1} + \partial_x^n \phi(z,x).(z - x)^n - n.\partial_x^{n-1}\phi(z,x).(z - x)^{n-1}$$

$$= \partial_x^n fz.(z - x)^{n-1} + \partial_x \left\{\partial_x^{n-1}\phi(z,x).(z - x)^n\right\}.$$

Hence, if A be the greatest value which $\partial_x^n fx$ can assume while x *increases* from x_1 to z,

$$A\,(z - x)^{n-1} + \partial_x \left\{\partial_x^{n-1}\phi(z,x).(z - x)^n\right\},$$

$$\text{or } \partial_x \left\{-\frac{A}{n}(z - x)^n + \partial_x^{n-1}\phi(z,x).(z - x)^n\right\},$$

or $\partial_x\, Fx$, (representing the quantity between the brackets by Fx) is not negative for any value of x from x_1 to z.

Assume $z - x_1 = mk$, and let P_0, P_1, P_2... represent the values of $\partial_x Fx$ corresponding to the values x_1, $x_1 + k$, $x_1 + 2k$,... of x; then, since

$$F\overline{x_1 + k} - Fx_1 = P_0.\frac{k}{1} + \text{\&c.}$$

$$F\overline{x_1 + 2k} - F\overline{x_1 + k} = P_1.\frac{k}{1} + \text{\&c.}$$

$$\dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots$$

$$Fz - F\overline{x_1 + (m - 1)k} = P_{m-1}.\frac{k}{1} + \text{\&c.}$$

$$\therefore\ Fz - Fx_1 = (P_0 + P_1 + P_2 + \dots + P_{m-1}).\frac{k}{1} + \text{\&c.};$$

and if m be sufficiently increased or k sufficiently diminished, the sign of the whole series will be the same with the sign of its first term. But the sign of this term is positive, since some of the quantities P_0, P_1, P_2, &c. are finite and positive and none are negative; therefore

$$Fz \text{ is } > Fx_1.$$

But since $\{\partial_z^{n-1}\phi(z,x)\}_{z=x} = \dfrac{\partial_x^n fx}{n}$, or is finite, (Art. 99),

$$Fz = 0, \text{ and } \therefore\ Fx_1 \text{ is negative.}$$

Hence $\dfrac{A}{n}.(z - x_1)^n$ is $> \{\partial_z^{n-1}\phi(z,x)\}_{x=x_1}.(z - x_1)^n$;

and $\dfrac{A\,(z - x_1)^n}{\underline{|n}}$ is $> \{\partial_z^{n-1}\phi(z,x)\}_{x=x_1}.\dfrac{(z - x)^n}{\underline{|n-1}}$

whatever be the value of z, if $> x_1$. And if $x_1 + h$ be substituted for z, the latter of the above quantities is the remainder (R) of the expansion of $f\overline{x_1+h}$ by Taylor's Theorem after the n first terms (Art. 99).

$$\therefore\ A.\frac{h^n}{\underline{|n}} \text{ is } > R.$$

Similarly, if B be the least value which $\partial_x^n fx$ can assume while x changes from x_1 to $x_1 + h$, it may be shewn that

$$B.\frac{h^n}{\underline{|n}} \text{ is } > R.$$

Hence the remainder of the series after the first n terms is between the greatest and least values which the first neglected term can assume while x changes from x_1 to $x_1 + h$. And the same may be shewn to be true when h is negative.

Ex. 1. Let $fx = \epsilon^x$, then $\partial_x^n fx = \epsilon^x$,

$$\therefore \text{ the first neglected term} = \epsilon^x.\frac{h^n}{\underline{|n}}.$$

$\therefore$ limits of the remainder after n terms are

$$\epsilon^{x_1}.\frac{h^n}{\underline{|n}} \text{ and } \epsilon^{x_1+h}.\frac{h^n}{\underline{|n}}.$$

Ex. 2. Let $fx = \sin x$, then

$$\partial_x^n fx = \sin \overline{n.\frac{\pi}{2} + x};$$

$\therefore$ the first neglected term $= \sin \overline{n.\frac{\pi}{2} + x}.\frac{h^n}{\lfloor n}$;

$\therefore$ limits of the remainder after n terms are the least and greatest values of this quantity for any value of x from x_1 to $x_1 + h$.

If h be not $< 2\pi$, the limits are

$$\frac{h^n}{\lfloor n} \text{ and } -\frac{h^n}{\lfloor n}.$$

If h be between 0 and π, the limits are

$$0 \text{ and } \pm\frac{h^n}{\lfloor n}.$$

V. *To approximate to the roots of equations by means of Taylor's Theorem.*

Let $fx = 0$ represent an equation, one of whose roots is known, by trial or otherwise, to exceed x_1 by a very small quantity. Then, if h be that quantity

$$f\overline{x_1 + h} = 0;$$

or, by Taylor's Theorem, if the equation be of n dimensions,

$$(fx)_{x=x_1} + (\partial_x fx)_{x=x_1}.\frac{h}{1} + (\partial_x^2 fx)_{x=x_1}.\frac{h^2}{\lfloor 2} + \dots\dots + Ch^n = 0;$$

and, since h is very small, its approximate value will be determined from

$$(fx)_{x=x^1} + (\partial_x fx)_{x=x^1}.h = 0;$$

$$\therefore\ h = -\left\{\frac{fx}{\partial_x fx}\right\}_{x=x_1}:$$

and the approximate value of the root is $x_1 - \left(\frac{fx}{\partial_x fx}\right)_{x=x_1}$.

Similarly, if x_2 represent this approximate value, another approximation will be obtained by taking $x_2 - \left(\frac{fx}{\partial_x fx}\right)_{x=x_2}$; and so on. And if each of these be nearer the true value than the preceding, the root may be determined to any degree of accuracy. For the limitations of this method see Hymers' *Theory of Equations*, Art. 123.

Ex. Required a root of the equation $x^3 - 3x + 1 = 0$.

It is found by trial that one of the roots lies between 1.5 and 2; therefore if fx represent $x^3 - 3x + 1$, an approximate value of x is

$$1.5 - \left\{\frac{fx}{\partial_x fx}\right\}_{x=1.5}.$$

$$\text{But } \partial_x fx = 3x^2 - 3, \quad \therefore \; (\partial_x fx)_{x=1.5} = 3.75,$$

$$\text{and } (fx)_{x=1.5} = -.125;$$

$$\therefore \text{ the first approximate value of } x = 1.5 + \frac{.125}{3.75}$$

$$= 1.533.$$

VI. *To transform a given equation into another of which the roots are greater or less than those of the former by a given quantity.*

Let $fx = 0$ represent the given equation of n dimensions; and let it be required to transform it into another whose roots are greater or less by the quantity h.

Assume $y = x - h$; then $x = h + y$, and $f\overline{h+y} = 0$; or, by Maclaurin's Theorem, since

$$(f\overline{h+y})_{y=0} = (fx)_{x=h},$$

$$(\partial_y f\overline{h+y})_{y=0} = (\partial_x fx)_{x=h},$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$(fx)_{x=h} + (\partial_x fx)_{x=h}\cdot\frac{y}{1} + (\partial_x^2 fx)_{x=h}\cdot\frac{y^2}{\underline{|2}} + \dots + (\partial_x^n fx)_{x=h}\cdot\frac{y^n}{\underline{|n}} = 0,$$

which is the transformed equation arranged according to ascending powers of y, each of whose roots is less than those of $fx = 0$ by the quantity h.

Similarly, if $y = x + h$, substituting $-h$ for h in the above equation, the transformed equation is obtained, each of whose roots is *greater* than those of $fx = 0$ by the quantity h.

Ex. Required to transform the equation

$$x^3 - 9x^2 + 7x + 12 = 0$$

into one whose roots shall be those of the proposed equation diminished by 3.

Assume $y = x - 3$; then

$$\begin{array}{ll} fx = x^3 - 9x^2 + 7x + 12, & \text{and } (fx)_{x=3} = -21; \\ \partial_x fx = 3x^2 - 18x + 7, & \therefore\ (\partial_x fx)_{x=3} = -20; \\ \partial_x^2 fx = 6x - 18, & \therefore\ (\partial_x^2 fx)_{x=3} = 0; \\ \partial_x^3 fx = 6, & \therefore\ (\partial_x^3 fx)_{x=3} = 6; \end{array}$$

But $(fx)_{x=3} + (\partial_x fx)_{x=3}\cdot\frac{y}{1} + (\partial_x^2 fx)_{x=3}\cdot\frac{y^2}{\underline{|2}} + (\partial_x^3 fx)_{x=3}\cdot\frac{y^3}{\underline{|3}} = 0,$

$\therefore\ y^3 - 20y - 21 = 0$, which is the required equation.

VII. *To expand* $f\overline{x+h}$. TAYLOR'S THEOREM. [Another proof.]

LEMMA. To prove that $\partial_x f\overline{x+h} = \partial_h f\overline{x+h}$.

$$\text{Since } \partial_x\overline{x+h} = 1 = \partial_h\overline{x+h},$$

$$\begin{aligned} \therefore\ \partial_x f\overline{x+h} &= \partial_{x+h} f\overline{x+h}.\partial_x\overline{x+h} \quad \text{(Art. 15)} \\ &= \partial_{x+h} f\overline{x+h} \\ &= \partial_{x+h} f\overline{x+h}.\partial_h\overline{x+h} \\ &= \partial_h f\overline{x+h}. \end{aligned}$$

Now, since $f\overline{x+h} = fx + \frac{f\overline{x+h} - fx}{h}.h$, let the coefficient of h be represented by P_1, so that

$$f\overline{x+h} = fx + P_1.h.$$

$$\text{Then } \partial_x f\,\overline{x+h} = \partial_x fx + \partial_x P_1.h,$$

$$\text{and } \partial_h f\,\overline{x+h} = P_1 + \partial_h P_1.h;$$

$$\therefore \text{ by Lemma, } P_1 + \partial_h P_1.h = \partial_x fx + \partial_h P_1.h,$$

$$\text{or, } P_1 = \partial_x fx + (\partial_x P_1 - \partial_h P_1).h$$

$$= \partial_x fx + P_2.h, \text{ if } P_2 \text{ represent } \partial_x P_1 - \partial_h P_1;$$

$$\therefore\ f\,\overline{x+h} = fx + \partial_x fx.h + P_2 h^2 \ldots\ldots\ldots\ldots(1).$$

$$\text{Again } \partial_x f\,\overline{x+h} = \partial_x fx + \partial_x^2 fx.h + \partial_x P_2.h^2,$$

$$\text{and } \partial_h f\,\overline{x+h} = \partial_x fx + \partial_x P_2.h^2 + 2P_2.h;$$

$$\therefore \text{ by Lemma, } 2P_2.h = \partial_x^2 fx.h + (\partial_x P_2 - \partial_h P_2).h^2,$$

$$\text{or } P_2 = \partial_x^2 fx.\frac{1}{2} + P_3.\frac{h}{2}, \text{ if } P_3 \text{ represent } \partial_x P_2 - \partial_h P_2;$$

$$\therefore\ f\,\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{\lfloor 2}} + P_3.\frac{h^3}{\underline{\lfloor 2}} \ldots\ldots\ldots\ldots(2).$$

Assuming that the law, thus indicated in the expansion as far as it has proceeded, holds for n terms, then

$$f\,\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\underline{\lfloor 2}} + \ldots\ldots$$

$$+ \partial_x^{n-1} fx.\frac{h^{n-1}}{\underline{\lfloor n-1}} + P_n.\frac{h^n}{\underline{\lfloor n-1}};$$

$$\therefore\ \partial_x f\,\overline{x+h} = \partial_x fx + \partial_x^2 fx.\frac{h}{1} + \partial_x^3 fx.\frac{h^3}{\underline{\lfloor 2}} + \ldots\ldots$$

$$+ \partial_x^n fx.\frac{h^{n-1}}{\underline{\lfloor n-1}} + \partial_x P_n.\frac{h^n}{\underline{\lfloor n-1}},$$

$$\text{and } \partial_h f\,\overline{x+h} = \partial_x fx + \partial_x^2 fx.\frac{h}{1} + \partial_x^3 fx.\frac{h^3}{\underline{\lfloor 2}} + \ldots\ldots$$

$$+ \partial_h P_n.\frac{h^n}{\underline{\lfloor n-1}} + n.P_n.\frac{h^{n-1}}{\underline{\lfloor n-1}};$$

$$\therefore \text{ by Lemma, } n.P_n = \partial_x^n fx + (\partial_x P_n - \partial_h P_n).h,$$

$$\text{or } P_n = \partial_x^n fx.\frac{1}{n} + P_{n+1}.\frac{h}{n};$$

$$\therefore f\,\overline{x+h} = fx + \partial_x fx.\frac{h}{1} + \partial_x^2 fx.\frac{h^2}{\lfloor\underline{2}} + \ldots + \partial_x^n fx.\frac{h^n}{\lfloor\underline{n}} + P_{n+1}.\frac{h^{n+1}}{\lfloor\underline{n}}.$$

Hence, if the law of the expansion hold for n terms, it holds also for $\overline{n+1}$; but it has been proved true as far as three, therefore it is true for four; and, if for four, for five; and so on; i.e., generally for any number.

EXAMPLES.

I. DIFFERENTIATION OF EXPLICIT FUNCTIONS OF ONE VARIABLE.

1. $\partial_x\{x^3(a+x)^2\} = (3a+5x)(a+x).x^2.$

2. $\partial_x\left(\dfrac{x^3}{a^3}+\dfrac{a^2}{x}\right) = 3\left(\dfrac{x}{a}\right)^2 - \left(\dfrac{a}{x}\right)^2.$

3. $\partial_x(\overline{a+x}.\sqrt{a-x}) = \dfrac{a-3x}{2\sqrt{a-x}}.$

4. $\partial_x\dfrac{x}{\sqrt{a-bx^2}} = \dfrac{a}{(a-bx^2)^{\frac{3}{2}}}.$

5. $\partial_x\dfrac{1+x^2}{1-x^2} = \dfrac{4x}{(1-x^2)^2}.$

6. $\partial\,\dfrac{x^n}{1+x^n} = \dfrac{nx^{n-1}.\partial x}{(1+x)^{n+1}}.$

7. $\partial_x\dfrac{(1+x)^3}{(1-x)^4} = \dfrac{(1+x)^2}{(1-x)^5}(7+x).$

8. $\partial_x\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1}{(1-x).\sqrt{1-x^2}}.$

9. $\partial_x\dfrac{\sqrt{a^2-x^2}}{x} = \dfrac{-a^2}{x^2.\sqrt{a^2-x^2}}.$

10. $\partial_x\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = -\dfrac{2}{x^3\sqrt{1-x^4}} - \dfrac{2}{x^3}.$

11. $\partial_\theta(\cos\theta + \sec\theta) = \sin\theta.(\tan\theta)^2.$

12. $\partial_x\dfrac{\sqrt[m]{(1+x^2)^2}}{\sqrt[n]{(1-x^2)^2}} = \dfrac{4}{mn}.\dfrac{\sqrt[m]{(1+x^2)^{2-m}}}{\sqrt[n]{(1-x^2)^{2-n}}}\{(m+n)x-(m-n)x^2\}.$

13. $\partial_x \dfrac{(x+3)^3}{(x+2)^2} = \dfrac{(x+3)^2}{(x+2)^3}.x.$

14. $\partial_x \dfrac{(x-1)^2}{(x+1)^3} = \dfrac{x-1}{(x+1)^4}.(5-x).$

15. $\partial_x \sin \overline{ax - bx^2} = (a - 2bx).\cos \overline{ax - bx^2}.$

16. $\partial_x (x^n.a^x) = x^{n-1}.a^x (x \log_\epsilon a + n).$

17. $\partial_x \{\epsilon^x.(x^3 - 3x^2 + 6x - 6)\} = \epsilon^x.x^3.$

18. $\partial_x \dfrac{\epsilon^x.x}{1-x} = \dfrac{\epsilon^x}{(1-x)^2}(1 + x - x^2).$

19. $\partial_x (x.\log_\epsilon x) = \log_\epsilon x + 1.$

20. $\partial_x \log_\epsilon \log_\epsilon x = \dfrac{1}{x.\log_\epsilon x}.$

21. $\partial_x \log_\epsilon \left(\dfrac{1}{x} + \dfrac{1}{x^2}\right) = -\dfrac{x+2}{x(x+1)}.$

22. $\partial_x \log_\epsilon \dfrac{1 + \sqrt{1-x}}{x} = -\dfrac{1}{x\sqrt{1-x^2}}.$

23. $\partial_x \log_\epsilon \dfrac{(x+4)^2}{x+2} = \dfrac{x}{x^2 + 6x + 8}.$

24. $\partial_x \log_\epsilon \dfrac{(x+2)^2}{\sqrt{(x+1)(x+3)^3}} = \dfrac{x}{(x+1)(x+2)(x+3)}.$

25. $\partial_x \left\{2\sqrt{\dfrac{a}{a+x}} + \log_\epsilon \dfrac{\sqrt{a^2 + ax} - a}{\sqrt{a^2 + ax} + a}\right\} = \dfrac{1}{x}\left(\dfrac{a}{a+x}\right)^{\frac{3}{2}}.$

26. $\partial_x \left\{\dfrac{1}{\sqrt{a^2 - b^2}}.\log_\epsilon \dfrac{\sqrt{a+b} + \sqrt{a-b}.x}{\sqrt{a+b} - \sqrt{a-b}.x}\right\} = \dfrac{2}{a + b - (a-b)x^2}.$

27. $\partial_x (x^n.\epsilon^{\log_\epsilon x}) = (n+1)\, x^{n-1}\, \epsilon^{\log_\epsilon x}.$

28. $\partial_x \log_\epsilon \sin x = \cot x.$

29. $\partial_x \log_\epsilon \sqrt{\dfrac{1 + \sin x}{1 - \sin x}} = \dfrac{1}{\cos x}.$

30. $\partial_x \cos \log_\epsilon x = \frac{1}{x} \sin \log_\epsilon \frac{1}{x}.$

31. $\partial_x \sqrt{\operatorname{cosec} x} = -\frac{\cos x}{2(\sin x)^{\frac{3}{2}}}.$

32. $\partial_x \sin^{-1} \frac{1 - x^2}{1 + x^2} = -\frac{2}{1 + x^2}.$

33. $\partial_x \cos^{-1}(4x^3 - 3x) = -\frac{3}{\sqrt{1 - x^2}}.$

34. $\partial_x \operatorname{cosec}^{-1} \frac{1}{2x\sqrt{1 - x^2}} = \frac{2}{\sqrt{1 - x^2}}.$

35. $\partial_x \left(\frac{1}{2\sqrt{ab}} \sin^{-1} \frac{2x\sqrt{ab}}{a + bx^2}\right) = \frac{1}{a + bx^2}.$

36. $\partial_x \tan^{-1} \frac{3x - x^3}{1 - 3x^2} = \frac{3}{1 + x^2}.$

37. $\partial_x \left(\frac{1}{\sqrt{a^2 - b^2}} . \cos^{-1} \frac{b + a\cos x}{a - b\cos x}\right) = \frac{1}{a + b\cos x}.$

38. $\partial_x \epsilon^{\sin^{-1} x} = \frac{\epsilon^{\sin^{-1} x}}{\sqrt{1 - x^2}}.$

39. $\partial_x \left(\frac{1}{4\sqrt{2}} \log_\epsilon \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1 - x^2}\right) = \frac{1}{1 + x^4}.$

40. $\partial\{x^m . (\log_\epsilon x)^n\} = x^{m-1} . (\log_\epsilon x)^{n-1} . \{m . \log_\epsilon x + n\}.$

41. $\partial_x x^{\frac{1}{x}} = \frac{x^{\frac{1}{x}}}{x^2} . \log_\epsilon \frac{\epsilon}{x}.$

42. $\partial_x^3 (ax^3 - 2bx^2 + cx) = 6a.$

43. $\partial_x^3 \frac{a^2}{a^2 + x^2} = \frac{24a^2x(a^2 - x^2)}{(a^2 + x^2)^3}.$

44. $\partial_x^4\left\{1-2\left(\sin\frac{x}{2}\right)^2\right\} = \cos x.$

45. $\partial_x^n(x.\epsilon^x) = (x+n).\epsilon^x.$

46. $\partial_x^{2n+1}(\sin x.\cos x) = (-1)^{n+1}.2^{2n}.\{(\sin x)^2-(\cos x)^2\}.$

47. $\partial_x^n\frac{1+x}{1-x} = 2.\frac{\underline{|n}}{(1-x)^{n+1}}.$

48. $$\partial_x^n\frac{\epsilon^{ax}}{x} = \frac{a^n\epsilon^{ax}}{x}\left\{1-\frac{n}{ax}+\frac{m(n-1)}{a^2x^2}-\frac{n(n-1)(n-2)}{a^3x^3}+\right.$$
$$\left.\ldots\ldots\pm\frac{\underline{|n}}{a^nx^n}\right\}.$$

49. $$\partial^3\frac{1}{p^m} = -\frac{1}{p^{m+3}}\{m(m+1)(m+2).(\partial p)^3-3m(m+1)p.\partial p.\partial^2 p$$
$$+mp^2.\partial^3 p\}.$$

50. $\partial_x^3 x^x = -x^x.\left\{\frac{1}{x^2}-\frac{3}{x}(1+\log_\epsilon x)-(1+\log_\epsilon x)^3\right\}.$

II. DIFFERENTIATION OF IMPLICIT FUNCTIONS OF ONE VARIABLE.

1. If $xy-ax^2+by^2=0$, $\partial_x y = -\frac{y-2ax}{x-2by}.$

2. If $\frac{x+\sqrt{a^2-u^2}}{a} = \log_\epsilon\frac{a+\sqrt{a^2-u^2}}{u}$, $\partial_x u = -\frac{u}{\sqrt{a^2-u^2}}.$

3. If $u=\cos^{-1}\frac{u}{x}$, $\partial_x u = \frac{u}{x(1+\sqrt{x^2-u^2})}.$

4. If $x = m.\tan\overline{z - nx}$, $\quad \partial_z x = \dfrac{m^2 + x^2}{m + n(m^2 + x^2)}$.

5. If $au^3 - xu - a = 0$, $\quad \partial_x^2 u = -\dfrac{2ux}{(3au^2 - x)^3}$.

6. If $ay^3 - bxy + 1 = 0$, $\quad \partial_a y = \dfrac{y^3}{bx - 3ay^2}$.

7. If $\sin\overline{a + y} = a.\sin x + b.\cos x$, $\quad \partial_x y = \dfrac{a.\cos x - b.\sin x}{\cos\overline{a + y}}$.

III. DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES.

1. If $u = axy + by^2$, then $\partial_{(x)} u = ay$, $\partial_{(y)} u = ax + 2by$.

2. If $u = a + \sqrt[3]{x^2 + y^2}$, then $\partial_{(x)} u = \dfrac{2x}{3(x^2+y^2)^{\frac{2}{3}}}$,

$$\partial_{(y)} u = \frac{2y}{3(x^2+y^2)^{\frac{2}{3}}}.$$

3. If $z = e(x^2 + y^2)^{\frac{1}{2}}$, then $\partial_{(x)} z = \dfrac{e^2 x}{z}$, $\partial_{(y)} z = \dfrac{e^2 y}{z}$.

4. If $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2 = 1$, then $\partial_{(x)} z = -\dfrac{c^2 x}{a^2 z}$,

$$\partial_{(y)} z = -\frac{c^2 y}{b^2 z}.$$

5. If $z = e.\cos^{-1}\dfrac{x}{a}$, and $z = e.\sin^{-1}\dfrac{y}{a}$, then

$$\partial_{(x)} z = -\frac{e}{y}, \quad \partial_{(y)} z = \frac{e}{x}, \quad \partial_{(x)}^2 z = -\frac{ex}{y^3}.$$

6. If $(x+y+z).\epsilon^z = c$, then $\partial x + \partial y + \partial z + (x+y+z).\partial z = 0$.

7. If $(x + y)^2.(y + z) = c$, then

$$2.\partial x.(y + z) + \partial y.(x + 3y + 2z) + \partial z.(x + y) = 0.$$

8. If $\dfrac{ay - bz}{cz - ax} = c$, then

$$\partial x(ay - bz) + \partial y(cz - ax) + \partial z(bx - cy) = 0.$$

IV. CHANGE OF THE VARIABLE OF DIFFERENTIATION.

1. Transform $a + b.\partial_x y + c.\partial_x^2 y$ into the equivalent expression which has y for the variable of differentiation.

2. Given $x = a.\text{vers}\,\theta$, and $y = a\theta + a.\sin\theta$, required $\partial_x y$.

3. Given $x.\partial_x y - ay = 0$, prove that $\partial_z y - ay = 0$, where $z = \log_\epsilon x$.

4. If $\partial_x^2 y + \dfrac{a + 2x}{1 + x^2}..\partial_x y + \dfrac{by}{(1 + x^2)^2} = 0$, and $\theta = \tan^{-1} x$,

then $\partial_\theta^2 y + a.\partial_\theta y + by = 0$.

5. Required $\partial_z s$, when $\partial_x s = \dfrac{\sqrt{1 - e^2 x^2}}{1 - x^2}$, and $z = \dfrac{x}{\partial_x s}$.

6. Given $R = \dfrac{\{1 + (\partial_x y)^2\}^{\frac{3}{2}}}{-\partial_x^2 y}$, and $x = r.\cos\theta$, $y = r.\sin\theta$, shew that

$$R = \frac{\{r^2 + (\partial_\theta r)^2\}^{\frac{3}{2}}}{r^2 + 2(\partial_\theta r)^2 - r.\partial_\theta^2 r}.$$

7. If $z = f(x, y)$ where both x and y are functions of two other variables r and θ, express $\partial_{(x)} z$ and $\partial_{(y)} z$ in terms of r, θ, $\partial_{(r)} z$ and $\partial_{(\theta)} z$; and, if $x = r.\cos\theta$, $y = r.\sin\theta$, prove that

$$x.\partial_{(y)} z + y.\partial_{(x)} z = \partial_{(\theta)} z.$$

8. Given $z = r.\sin\phi$, $y = r.\cos\phi.\sin\theta$, $x = r.\cos\phi.\cos\theta$, transform the expression

$$1 + (\partial_{(x)} z)^2 + (\partial_{(y)} z)^2$$

into the equivalent expression involving only r, θ, ϕ, and the partial differentials of r with regard to θ and ϕ.

V. ELIMINATION OF CONSTANTS AND FUNCTIONS.

1. If $y + ax + b = 0$, then $\partial_x y + a = 0$, and $x.\partial_x^2 y = 0$.

2. If $x^2 + y^2 = cx$, then $y - x.\partial_x y = x\sqrt{1 + (\partial_x y)^2}$.

3. If $x\epsilon^{\frac{x}{y}} = c$, then $xy + y^2 = x^2.\partial_x y$.

4. If $x^2 - 2ay - a^2 - b = 0$, then $x - a.\partial_x y = 0$, and

   $\partial_x y.\sqrt{y^2 + x^2 - b} - y.\partial_x y - x = 0$. Also $x.\partial_x^2 y = \partial_x y$.

5. If $(x - a)^2 + (y - b)^2 = r^2$, then $\dfrac{\{1 + (\partial_x y)^2\}^3}{(-\partial_x^2 y)^2} = r^2$.

6. If $\log_\epsilon \dfrac{x}{a} = \sin^{-1}\dfrac{y}{x}$, then $x.\partial_x y - y = \sqrt{x^2 - y^2}$.

7. If $y = \dfrac{\epsilon^{2x} + 1}{\epsilon^{2x} - 1}$, then $\partial_x y = 1 - y^2$.

8. If $y = \epsilon^x.\cos x$, then $\partial_x^2 y - 2.\partial_x y + 2y = 0$.

9. If $(1 - x).y = ax^2 + bx + c$, then $(1 - x).\partial_x^3 y - 3.\partial_x^2 y = 0$.

10. If $u = m\sin x + n\cos x$, then $\partial_x^2 u + u = 0$.

11. If $y^2 = mx + nx^2$, eliminate m and n.

12. If $y = \epsilon^x.(a + bx) + c.\cos x + d.\sin x$, then

    $y - 2.\partial_x y + 2.\partial_x^2 y - 2.\partial_x^3 y + \partial_x^4 y = 0$.

13. If $z = x^n.\phi y$, then $\partial_{(x)} z = \dfrac{nz}{x}$.

14. If $\dfrac{y - b}{z - c} = F\dfrac{x - a}{z - c}$, then $z - c = \partial_{(x)} z.(x - a) + \partial_{(y)} z.(y - b)$.

15. If $z = \dfrac{x}{a} + \phi\,\overline{ay - bx}$, then $a.\partial_{(x)} z + b.\partial_{(y)} z = 1$.

16. If $z = \dfrac{y^2}{3x} + \phi\,\overline{xy}$, then $\partial_{(y)}z = \dfrac{x}{y}.\partial_{(x)}z + \dfrac{y}{x}$.

17. If $x^2 + y^2 + z^2 = \phi\,\overline{z + ax + by}$, then

$$(y - bz).\partial_{(x)}z - (x - az).\partial_{(y)}z = bx - ay.$$

VI. FUNCTIONS UNDER THE FORM $\frac{0}{0}$, $0 \times \infty$, $\frac{\infty}{\infty}$.

1. $\left\{\dfrac{x^3 + 5ax^2 - 4a^2x - 2a^3}{x^2 - a^2}\right\}_{x=a} = \dfrac{9a}{2}$.

2. $\left\{\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}\right\}_{x=a} = \dfrac{1}{\sqrt{2a}}$.

3. $\left\{\dfrac{5 - 2x^2 - \sqrt{8x^2 + 1}}{x^2 - 1}\right\}_{x=1} = -\dfrac{10}{3}$.

4. $\left\{\dfrac{\sqrt{2a^3x - x^4} - \sqrt{ax^3}}{a - \sqrt{ax}}\right\}_{x=a} = 5a$.

5. $\left\{\dfrac{\epsilon^x - \epsilon^{\sin x}}{x - \sin x}\right\}_{x=0} = 1$.

6. $\left\{\dfrac{x^x - x}{1 - x + \log_e x}\right\}_{x=1} = -2$.

7. $\left\{\left(\sec\dfrac{\pi x}{2}\right)^2.\text{versin}\,2\pi x\right\}_{x=1} = 8$.

8. $\left\{2^x.\tan\dfrac{a}{2^x}\right\}_{x=\infty} = a$.

9. $\left\{\dfrac{\cos^{-1}(1 - x)}{\sqrt{2x - x^2}}\right\}_{x=0} = 1$.

10. $\left\{\dfrac{a^{a-n} - x^{x-n}}{a - x}\right\}_{x=a} = a^{a-n}.\left\{\dfrac{a-n}{a} + \log_e a\right\}$.

11. $$\left\{\frac{\tan\dfrac{\pi x}{2a}}{\dfrac{x^2}{a(x^2-a^2)}}\right\}_{x=a} = -\frac{4a}{\pi}.$$

12. $$\left\{\frac{x^{n+1}-a^{n+1}}{n+1}\right\}_{n=-1} = \log_e \frac{x}{a}.$$

13. If $(y^2-x^2)^2 = x^3 - 2axy^2$, $(\partial_x y)_{\substack{x=0\\y=0}} = \pm\dfrac{1}{\sqrt{2a}}$ and ∞.

14. If $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$, $(\partial_x y)_{\substack{x=\pm a\\y=0}} = \pm\sqrt{\dfrac{2}{3}}$,

and $(\partial_x y)_{\substack{x=\ 0\\x=-a}} = \pm\sqrt{\dfrac{4}{3}}$.

VII. EXPANSIONS.

By Maclaurin's Theorem.

1. $$(a+bx+cx^2)^{\frac{1}{2}} = a^{\frac{1}{2}}\left\{1 + \frac{b}{2a}.\frac{x}{1} - \frac{4ac-b^2}{2^2.a^2}.\frac{x^2}{1.2} + \frac{3b(4ac-b^2)}{2^3.a^3}.\frac{x^3}{1.2.3} - \&c.\right\}$$

2. $$\operatorname{Sin}^{-1}\frac{x}{a} = \frac{x}{a} + \frac{1}{2}.\frac{x^3}{3a^3} + \frac{1.3}{2.4}.\frac{x^5}{5a^5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7a^7} + \ldots\ldots$$

3. $$\frac{a+x}{b+x} = \frac{a}{b} - (a-b).\left\{\frac{x}{(b+x)^2} + \frac{x^2}{(b+x)^3} + \frac{x^3}{(b+x)^4} + \ldots\right\}$$

4. If $ay^3 - bxy + 1 = 0$, expanding y in powers of a,

$$y = \frac{1}{bx} + \frac{1}{b^4x^4}.a + \frac{3}{b^7x^7}.a^2 + \frac{12}{b^{10}x^{10}}.a^3 + \ldots\ldots$$

5. If $y^3 - 3axy + x^3 - 0$, expanding y in powers of a,

$$y = -x - a + \frac{1}{3x^2}.a^3 - \frac{1}{3x^3}.a^4 + \ldots\ldots$$

6. Expand by Maclaurin's Theorem $\frac{x^2}{(x-1)^2}$ in negative powers of x.

7. Given $mu^3 - ux - m = 0$, shew by Maclaurin's Theorem that

$$u = 1 + \frac{x}{3m} - \frac{x^3}{3^4 m^3} + \frac{x^4}{3^5 m^4} - \&c.$$

By Taylor's Theorem,

8. $\tan \overline{x+h} = \tan x + (\sec x)^2 . \frac{h}{1} + 2 . \sin x . (\sec x)^3 . \frac{h^2}{\lfloor 2} + \&c.$

9. $\sin^{-1} \overline{x+h} = \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} . \frac{h}{1} + \frac{x}{(1-x^2)^{\frac{3}{2}}} . \frac{h^2}{\lfloor 2} + \ldots$

10. Apply Lagrange's Theorem to determine approximately one of the roots of the equation

$$x^3 - 5x + 7 = 0.$$

11. If $x - a + bx^n = 0$, by Lagrange's Theorem,

$$x^m = a^m + ma^{m+n-1}b + m . \frac{m+2n-1}{2} a^{m+2n-2} . b^2 + \&c.$$

12. If $(1-e)\left(\tan\frac{v}{2}\right)^2 = (1+e)\left(\tan\frac{u}{2}\right)^2$, and $m = u - e . \sin u$, shew that

$$v = m + \left(2e - \frac{e^3}{4}\right) . \sin m + \frac{5e^2}{4} . \sin 2m + \frac{13e^3}{12} . \sin 3m + \&c.$$

VIII. MAXIMA AND MINIMA.

1. The maximum of $3a^2x^3 - b^4x + c^5$ is when $x = -\frac{b^2}{3a}$; and the minimum when $x = \frac{b^2}{3a}$.

2. The maximum of $\frac{\epsilon^x}{\sin \overline{a-x}}$ is when $x = \frac{\pi}{4} + a$.

3. Find the maxima and minima of $x^3 - 2x^2 + x + 4$; and shew that they are not the *greatest* and *least* values of the function.

4. If $u = \frac{x}{\log_\epsilon x}$, $u_{x=\epsilon}$ is a minimum.

5. If $y = x + \sqrt[3]{a^3 - x^3}$, $y_{x=\frac{a}{\sqrt[3]{2}}}$ is a maximum.

6. $\sqrt{a^2 + x^2} + \sqrt{a^2 + \left(\frac{ab}{x}\right)^2}$ has a maximum when $x = -\sqrt{ab}$, and a minimum when $x = \sqrt{ab}$.

7. If $u = \frac{x^2 - x + 1}{x^2 + x - 1}$, $u_{x=0}$ is a maximum, and $u_{x=2}$ a minimum.

8. $\sin x . \operatorname{vers} x$ has a maximum when $x = \frac{2\pi}{3}$.

9. Draw the least quadrilateral figure circumscribing a given circle.

10. Draw the shortest line touching a given ellipse and terminated by the tangents at the extremities of the axes of the ellipse.

11. Of all the cones cut out of a given sphere determine that which has the greatest convex surface.

12. A cylindrical mug of given small thickness is required to be of a given capacity, find the least quantity of material of which it can be made.

13. If $u = x^3 + y^3 - 3axy$, $u_{\substack{x=a \\ y=a}}$ is a minimum.

14. Required the maximum of $(x+1).(y+1).(z+1)$, subject to the condition $a^x.b^y.c^z = A$.

Ans. $\dfrac{(\log_\epsilon \sqrt[3]{Aabc})^3}{\log_\epsilon a.\log_\epsilon b.\log_\epsilon c}$.

15. Find the maxima and minima of

$$a.\cos x + b.\cos y + c.\cos z,$$

when x, y, z are connected together by the equation

$$(\cos x)^2 + (\cos y)^2 + (\cos z)^2 = 1.$$

IX. CURVES.

1. Required the direction of curvature at any point of the curve whose equation is $y = ma.\text{vers}^{-1}\dfrac{x}{a} + \sqrt{2ax - x^2}$; and also the subtangent for the same point.

2. Find the equation to the tangent at any point of the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; and thence shew that the portion of the tangent, intercepted between the axes of co-ordinates, is the same for all points.

3. Determine the tangent at any point of the curve whose equation is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$, and prove that it coincides with a normal to the curve whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Determine the tangent to a curve, which being terminated by the axes of rectangular co-ordinates (1) cuts off the triangle of least area; and (2) cuts off the triangle which generates the least cone by revolution about the axis of abscissas.

Ans. The triangle is least when the tangent is bisected at the point of contact.
The cone is least when the tangent is trisected at the point of contact.

5. The curve whose equation is $y = \frac{x}{1+x^2}$ is concave for all values of x from 0 to $\pm\sqrt{3}$, and convex from $\pm\sqrt{3}$ to $\pm\infty$.

6. If $x^3 + bx^2 + a^2y = 0$ be the equation to a curve, the co-ordinates of a point of contraflexure are

$$x = -\frac{b}{3}, \quad y = -\frac{2b^3}{27a^2}.$$

7. If $ay = \pm x\sqrt{2ax - x^2}$ be the equation to a curve, there are maxima when $x = \frac{3a}{2}$, and points of contraflexure when $x = \frac{a}{2}\{3 - \sqrt{3}\}$.

8. In the curve, whose equations are $x=a\theta$, $y=a(1+\cos\theta)$, there is a point of contraflexure when $x = \frac{1}{2}\pi a$ and $y = a$.

9. In the curve whose equation is $y - b = 2(x-a)^5$, there is a point of contraflexure when $x = a$.

10. If $y^4 + 2axy^2 - ax^3 = 0$ be the equation to a curve, there is a multiple point at the origin, where *three* branches meet, one of which intersects the axis of x at right angles, and the other two intersect it at angles $\tan^{-1}\frac{1}{\pm\sqrt{2}}$.

11. If $ay^2 - x^3 + bx^2 = 0$ be the equation to a curve, the origin is an *isolated* point.

12. The curve whose equation is

$$a(y^2 - b^2) = \pm(x^2 - c^2).\sqrt{c^2 - 2x^2}$$

has four isolated points, corresponding to $x = \pm c$ and $y = \pm b$.

13. The semicubical parabola, whose equation is $ay^2 = x^3$, has a "cusp of the 1st species" at the origin.

14. There are cusps in the curves represented by each of the following equations:—

(1) $(y-2)^3 = (x-1)^2$ (3) $a^4y^2 = (x^2-a^2)^3$.

(2) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$, (4) $x^4-ax^2y-axy^2+\frac{1}{4}a^2y^2=0$.

15. Required the Radius of Curvature at the vertex of an ellipse, hyperbola, or parabola.

16. Required the Radius of Curvature at any point of the catenary, whose equation is $2y = a.\{\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}}\}$.

17. Find the Radius of Curvature at any point of the curve given by the equations

$$x = a(1-\cos\theta), \quad y = a(\theta+\sin\theta).$$

18. Find the co-ordinates of the point in which the parabola intersects its Evolute.

19. The equation to the Evolute of the curve represented by $xy = a^2$ is $(\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$.

20. Shew that the Evolute of a Cycloid is an equal Cycloid; and find its position.

21. If x, y be the co-ordinates of any point of a plane curve, and r the distance of the point from the origin, prove that the co-ordinates of the corresponding point of the Evolute are

$$\frac{1}{2}.\frac{\partial_y^2(r^2)}{\partial_y^2 x} \text{ and } \frac{1}{2}.\frac{\partial_x^2(r^2)}{\partial_x^2 y}.$$

22. If X, Y be the co-ordinates of the Centre of Curvature at the point (x, y) of the curve whose equation is $Ax^2+By^2+C=0$, shew that

$$X = \frac{A(A-B)}{BC}x^3, \text{ and } Y = \frac{B(B-A)}{AC}y^3.$$

23. The Chord of Curvature, at any point (x, y) of a Curve, drawn through any other point (α, β) is equal to

$$\frac{2\{1 + (\partial_x y)^2\}.\{y - \beta - \partial_x y(x - \alpha)\}}{\partial_x^2 y.\sqrt{(x - \alpha)^2 + (y - \beta)^2}}.$$

24. If $y = \dfrac{3x^2}{\sqrt{x^4 + a^4}}$ be the equation to a curve, $Y = 3x$ is the equation to its asymptote.

25. If $y = \dfrac{4x^3 - 2x^2}{x^2 + a^2}$ be the equation to a curve, $Y = 4x - 2$ is the equation to its asymptote.

26. If $x^3 + a^2 y = 2x^2 y$ be the equation to a curve, $Y = \frac{1}{2}x$ is the equation to its asymptote.

27. If $ax^2 = y^3 + a^3$ be the equation to a curve, $Y^3 = ax^2$ is the equation to its asymptote.

28. If $x^2 y - 2ay^2 - a^2 x = 0$ be the equation to a curve, $x^2 = 4aY$ and $Y = 0.x$ are equations to its asymptotes.

29. Prove that the tangent at any point of a conic section makes equal angles with the focal distances.

30. If $r = a\theta$ be the polar equation to a curve, shew that the subtangent at any point $(r, \theta) = \dfrac{r^2}{a}$.

31. If $r.\cos\theta + a.\cos 2\theta = 0$, the subtangent $= a$.

32. If $r = \sec\dfrac{\theta}{n}$, the subtangent $= \dfrac{nr}{\sqrt{r^2 - 1}}$.

33. The curve whose equation is $r.\cos\theta = a.\cos 2\theta$ is at every point concave towards the pole.

34. If $r^2 = a^2.\cos 2\theta$ be the polar equation to a curve, the Radius of Curvature at any point $(r, \theta) = \dfrac{a^2}{3r}$.

35. If $r = a(1 + \cos\theta)$ be the equation to a curve, the Radius of Curvature at any point $(r, \theta) = \frac{2}{3}\sqrt{2ar}$.

36. If $r = \dfrac{a\theta^2}{\theta^2 - 1}$, the curve has two rectilinear asymptotes at the distance $\dfrac{a}{2}$ from the focus, and making angles 1 and -1 with the prime radius.

37. Determine the form at the origin of the curve whose equation is $y^3 = ax^2 - x^3$.

38. Approximate to the general form of the curve whose equation is $(x^2 + y^2)^{\frac{3}{2}} = 2axy$.

39. A straight line of given length, having its extremities in the axes of x and y, always touches the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

40. A series of Ellipses are described from the equation $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{\beta}\right)^2 = 1$, in which a and β may have any values subject to the condition $a^n + \beta^n = c^n$; find the curve which touches all the Ellipses.

For more Examples the Student is referred to the extensive Collection by Mr Hind, entitled "*A Digested Series of Examples in the Applications of the Principles of the Differential Calculus.*"

THE END.